A Review of the Relativistic Static Fluid Sphere

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1 Introduction

One of the largest concentrations of literature within the area of exact solutions to Einstein's field equations is that of static, spherically symmetric perfect fluid (hereafter SSSPF) solutions. The reasons for such an extensive literature may be because it is comparatively easy to find a physical application for such solutions: provided the solutions satisfy regularity and reality conditions (for which we give some definitions in §3), they can provide valid models for neutron star interiors, where matter is sufficiently concentrated to warrant a general relativistic description of the fluid involved. An alternative explanation for the large number of published solutions could be that the high symmetry involved reduces the problem to the solution of two ordinary differential equations in two functions: one second order, the other first order. By suitable choice of one of these functions all that needs to be done is to solve a linear first-order ODE.

Whatever the reason for the number of solutions, one thing is clear — solutions have been (and are being) rediscovered with authors seemingly unaware that their solution has appeared many years before. The most striking example of this to date must be that of a solution due to Wyman for which we have found *seven* independent authors. With this in mind, one of the purposes of this review is to provide a central reference for people working in this area and to help reduce the amount of duplication in the field. In addition to this, using the objective criteria of regularity and physicality, we hope to highlight what we believe are the most interesting SSSPF solutions.

The form of this review is as follows: §**??** is an introduction to the area of SSSPF solutions, providing general background and describing coordinate systems in use; §3 is a summary of regularity and physicality conditions which are used later in the review and also give a summary of the relationships between regularity and metric components in various coordinate systems; §**??** contains reviews of all the papers known and available to the authors at the time of writing; §**??** contains a set of table of ansätze used for producing solutions, and equations in §**??** where a complete description of the associated solution may be found; finally §**??** picks out the more useful SSSPF solutions and studies them in more detail.

Those wishing to find out whether a solution is already known should probably best begin by consulting the tables in §**??** to see whether the form of the metric coefficients or density distribution they have chosen is represented there. Solutions which are not elementary may perhaps be found by looking through the ansätze used in §**??**.

2 Background

In what follows geometrised units are used with $8\pi G = c = 1$. The sign conventions used are those of the Landau-Lifshitz timelike convention [70]. The metric of a static spherically symmetric spacetime can be taken to be

$$ds^2 = e^{2\nu(r)}dt^2 - y^{-2}dr^2 - R^2(r)d\Omega^2 \tag{2.1}$$

where $d\Omega^2$ is the metric on the 2-sphere and the radial coordinate r is free to be transformed [**?**]. Denoting a derivative with respect to r by a prime, Einstein's field equations for a static fluid sphere may be written as

$$\rho = \frac{1}{R^2} - y^2\left(\frac{2R''}{R} + \frac{R'^2}{R^2} + 2\frac{y'}{y}\frac{R'}{R}\right) \tag{2.2}$$

$$p = y^2\left(\frac{R'}{R} + 2\nu'\right) - \frac{1}{R^2} \tag{2.3}$$

$$p \quad = \quad y^2 \left[\frac{R''}{R} + \left(\nu' + \frac{y'}{y} \right) \left(\nu' + \frac{R'}{R} \right) + \nu'' \right] \tag{2.4}$$

where we have assumed that the cosmological constant, Λ, vanishes. If this generalisation is required, the transformations

$$p \to p - \Lambda, \qquad \rho \to \rho + \Lambda$$

recover the full Einstein equations. Equations (2.2)–(2.4) have been rearranged in many ways. For example

$$p' = -(\rho + p)\nu' \tag{2.5}$$

is the contracted Bianchi identity. Also p can be eliminated from (2.4) and (2.4) to form the equation of pressure isotropy, which links $v(r)$, $y(r)$ and $R(r)$. This equation is the starting point for most of the solutions in the literature (see, eg, [**?**]):

$$(y^2)' \left(\frac{R'}{R} + \nu' \right) + 2y^2 \left(\frac{R''}{R} - \frac{R'^2}{R^2} - \nu' \frac{R'}{R} + \nu'' + \nu'^2 \right) + \frac{2}{R^2} = 0. \tag{2.6}$$

The equation is first-order and linear in y^2, and second-order, linear and homogeneous in e^ν, so it is not surprising that this is a popular starting place.

Before describing some of the particular coordinate systems used, we first define some physically relevant quantities in the general coordinate system (2.1).

If a function $m(r)$ is defined as

$$m(r) \equiv \frac{1}{2} R \left(1 - r'^2 y^2 \right), \tag{2.7}$$

then equation (2.2) may be written as

$$\rho = \frac{2m'}{R^2 R'}. \tag{2.8}$$

Setting $r = R$ then gives

$$m(R) = \frac{1}{2} \int_0^R \rho R^2 dR \tag{2.9}$$

leading to the interpretation of $m(R)$ as the mass inside a radius R (see box (23.1) of [74]). Using (2.4) and (2.7) leads to

$$\frac{\nu'}{R'} = \frac{2m + pR^3}{2R(R - 2m)}, \tag{2.10}$$

which, when used with (2.5) gives the Oppenheimer-Volkoff equation [80]

$$\frac{dp}{dR} = -\frac{(p + \rho)(2m + pR^3)}{2R(R - 2m)}. \tag{2.11}$$

Another useful physical parameter is the redshift of electromagnetic radiation emitted with wavelength λ_e and observed at a wavelength λ_o. In terms of the coordinate ν for a spherical distribution of matter this is

$$z \equiv \frac{\lambda_e}{\lambda_o} - 1 = e^{\nu_o - \nu_e} - 1. \tag{2.12}$$

For an observer sufficiently far from the sphere to be approximated to flat space, the observed redshift of light received from a point r_e in the sphere is

$$z_e = e^{-\nu_e} - 1. \tag{2.13}$$

Theoretically, both the mass and redshift can be observed far from the sphere, yet give information about the metric coefficients at points within the sphere.

Turning now to specific coordinate systems, various interpretations of spherical symmetry have favoured certain coordinate systems, and hence particular choices for the radial coordinate r in (2.1). The most common coordinate system — so common that it has sometimes been referred to as "canonical" or "standard" coordinates [?] — is what we shall term Schwarzschild coordinates. These are defined by taking R to be the radial coordinate (the radius of the 2-sphere with proper area $4\pi R^2$). Setting $R = r$ and defining $e^{-2\lambda} \equiv y^2$ in (2.1) the metric becomes

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\Omega^2. \tag{2.14}$$

Schwarzschild coordinates account for roughly 55% of the work in fluid spheres.

The second most popular coordinate system used in fluid sphere work, accounting for approximately 35% of papers, is isotropic coordinates. In this coordinate system, the spacelike part of the metric is Galilean. This is achieved by defining

$$r_{iso} \equiv \exp \int \frac{1}{Ry} dr, \tag{2.15}$$

whence, on dropping the subscript on r, and defining $e^{\mu} \equiv y^{-1}$, the line element (2.1) becomes

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\mu(r)}(dr^2 + r^2 d\Omega^2). \tag{2.16}$$

Of other coordinate choices used, accounting for the remaining 10% of papers, the only one to have been frequently used is Polar Gaussian coordinates. These were first used by Synge [87], but also appear in work by Ellis et al [31], Collins [?],[?], Hojman & Santamarina [42] and Berger & Hojman [6]. The choice of $y^2 = 1$ in (2.1) results in

$$ds^2 = e^{2\nu(r)}dt^2 - dr^2 - R^2(r)d\Omega^2. \tag{2.17}$$

Buchdahl has used coordinates (which we shall call Buchdahl's coordinates) defined by $y^2 = e^{2\nu}$, giving

$$ds^2 = e^{2\nu(r)}dt^2 - e^{-2\nu(r)}dr^2 - R^2(r)d\Omega^2. \tag{2.18}$$

A number of coordinate systems have also been used to study vacuum solutions but do not seem to have been used for SSSPF solutions. These include Eddington-Finkelstein coordinates with

$$u \equiv t - \int \frac{dr}{ye^{\nu}}, \quad ds^2 = e^{2\nu(r)}du^2 + 2du dr - R^2(r)d\Omega^2, \tag{2.19}$$

used by Eddington [?] and Finkelstein [34] in their studies of the Schwarzschild solution; half-null coordinates used by Kruskal [52] and Szekeres [88] in their work on event horizons, where

$$ds^2 = F(r)du dv - R^2(r)d\Omega^2, \tag{2.20}$$

and $u = u(r,t)$, $v = v(r,t)$ and finally the isothermal coordinates of Synge [87]:

$$y^2 = e^{-2\nu}, \quad ds^2 = e^{2\nu(r)}(dt^2 - dr^2) - R^2(r)d\Omega^2. \tag{2.21}$$

3 Regularity and Physicality

In this section we define a set of physical conditions which are demanded to hold in order that a solution be termed realistic. We also summarise the known characteristics of SSSPF solutions in order that they be regular at the centre of the sphere. What makes a solution realistic is somewhat subjective, and will depend from author to author on which energy conditions are considered inviolable, whether or not ultrabaric matter (where $p > \rho$) should be permitted, etc. We shall call a solution physically realistic if it meets the following conditions:

$$\rho \geq 0, \tag{3.22}$$

$$p \geq 0, \tag{3.23}$$

$$\rho \geq p, \tag{3.24}$$

$$v_s \equiv \frac{dp}{d\rho} \geq 0, \tag{3.25}$$

$$v_s \leq 1. \tag{3.26}$$

It is thus seen that we impose the weak and dominant energy conditions, but not the strong energy condition. We also demand that the fluid is stable The meaning of (3.26) is often disputed (see for example [?] and [?]), however it is the authors' belief that the present consensus of opinion is that (3.25) and (3.26) hold, certainly in the vast majority of cases.

In order that the fluid be stable against radial perturbations we also impose conditions on the radial variations of p and ρ: specifically p must decrease outwards and ρ must not increase outwards. In other words

$$\frac{dp}{dR} < 0, \tag{3.27}$$

$$\frac{d\rho}{dR} \leq 0. \tag{3.28}$$

Having defined what we mean by "realistic", we now look at a less restrictive but nevertheless useful criterion which we shall call "physical regularity". A solution will be called physically regular at a point if both ρ and p are finite there. The question of the relationship between physical regularity at the centre of a SSSPF distribution in various coordinates systems has been dealt with by many authors: Kuchowicz [?] seems to have been the first to analyse this problem in Schwarzschild coordinates, though Leibovitz [71] was not far behind; Glass and Goldman [39] studied this problem in isotropic coordinates and Korkina [45] used a formalism which is easily adapted to Schwarzschild coordinates.

The main result we shall use is for a metric in Schwarzschild coordinates: for such a metric to be physically regular at the centre, the metric coefficients g_{00} and g_{11} must be of the form

$$g_{00} = k\left(\frac{c}{r} + 1 + ar^{\beta} + \cdots\right), \quad g_{11} = \frac{c}{r} + 1 + br^{\alpha} + \cdots, \tag{3.29}$$

where a, b, c, k α and β are all constants with $\alpha, \beta > 2$. When $c \neq 0$ then $a = b$, with the consequence that ρ_0 and p_0 cannot be simultaneously positive. When $c = 0$, if both α and $\beta > 2$ then both p_0 and ρ_0 are zero. If only $\alpha > 2$ then $\rho_0 = 0$. If only $\beta > 2$, then gain it is impossible to have both ρ_0 and p_0 simultaneously positive.

In addition to a solution being physically regular and realistic, we also demand that the solution's geometry be regular. By this we mean the the geometry should be smooth and locally Minkowskian at all points. In particular, at the centre of the sphere we may use (??) to derive

$$R_0 = 0 \quad \left(\frac{m}{R^2}\right)_0 = 0. \tag{3.30}$$

We may integrate the second equation near the centre to obtain

$$m_0 \approx \frac{\rho_0 R^3}{6}. \tag{3.31}$$

On page 269 of Synge [87] it is noted that the conditions for smoothness at the surface of a sphere have "caused considerable confusion" and most papers listed in §?? gloss over this subject. Though it is dangerous to follow intuition in general relativity, we present here an intuitive treatment of this problem which we believe to be physically sound.

For a fluid sphere of finite extent there must be no singularity at the surface of the sphere, r_s. This is usually taken to mean that the metric coefficients ν, y and R and the mass m and pressure p are all continuous across the surface (see, for example, Bayin [?], Tooper [90] and Weinberg [93]). These requirements are intuitively obvious — a discontinuous metric coefficient would lead to a singular surface; a discontinuity in p cannot exist meaningfully in a fluid; and the total mass, $M = m_s$, of the sphere must be the same to an observer outside the sphere as one on the surface.

Equation (2.7) then implies that R'_s is continuous, and from (2.4) ν'_s must also be continuous. So at the surface

$$\nu(r),\ y(r),\ R(r),\ \nu'(r),\ R'(r) \text{ are continuous and } p(r) = 0. \tag{3.32}$$

The final condition follows since $p =$ outside the sphere. In practice this is used to define the sphere's surface $r = r_s$. The continuity of $m(r)_s$ follows from (2.7) and (3.32). Note that there is *no* restriction on the continuity of ρ_s or $y'(r)_s$ — they may be discontinuous. The special case where $\rho_s =$ has been singled out by Buchdahl [12] and termed a "gaseous" sphere.

Most work on surface conditions (see [35]–[37], [72], [76], [?]) uses coordinates in which the metric coefficients must be of class c^1 across $r = r_s$. Such coordinates are termed *admissible* coordinates. From (3.32) we see that most of the coordinate systems are not automatically admissible, since $y(r)_s$ may be discontinuous. However there are exceptions: Polar Gaussian coordinates (??) have $y'(r) = 0$ everywhere, and isotropic coordinates (??). In the special case of a gaseous sphere, Schwarzschild coordinates are also admissible, since (2.2) tells us

$$\rho_s = 0 \Rightarrow \left(\frac{R''}{R} + \frac{y'}{y}\frac{R'}{R}\right)_s \quad \text{is continuous,} \tag{3.33}$$

which, for Schwarzschild coordinates with$r = R$ and $R'' = 0$, gives

$$\rho_s = 0 \Rightarrow y'(r)_s \text{ in Scwarzschild coordinates is continuous,} \tag{3.34}$$

a result which does not seem to have been explicitly stated previously. Weinberg [93] shows that

$$M < \frac{4}{9}R_s,$$

so that matching e^ν at the surface with the Schwarzschild solution, we have

$$\frac{1}{3} < (e^\nu)_s < 1,$$

which, combined with (2.13) leads to the condition that the surface redshift must obey

$$z_s < 2 \tag{3.35}$$

4 Exact Solutions

In this section we review briefly some 60 key papers spanning the period from 1916 to 1992 which discuss SSSPF solutions in closed form. To give some sense of order of discovery, the papers are generally presented in chronological order, with slight deviations to gather related papers. The review is critical in the sense that many misprints and mistaken conclusions are highlighted and (we hope) corrected. In most cases this will be the first time that such explicit corrections are made, since the only previous review of this are of which we are aware is the section of Kramer et al [47], which mentions less than 25 papers, and with few details.

With the exception of v_s, used conventionally to denote the speed of sound, throughout this section the subscripts 0 and s will be used to denote values of quantities at the centre ($R = 0$) and surface ($p = 0$) of fluid spheres respectively. The mass of the sphere as measured at infinity will be denoted by M rather than m_s.

Several text-books, though they do not give reviews of the area, give a detailed treatment of fluid spheres. Among these are Adler et al [3], Misner et al [74], Weinberg [93] and Schutz [83]. They will not be mentioned further in this review.

Though not strictly an SSSPF solution, we must mention the unique, static, spherically symmetric vacuum solution — the Schwarzschild exterior solution, since we shall refer to it for matching reasons. For a mass M, the surrounding spacetime is described in Schwarzschild coordinates by [84]

$$e^{2\nu} = e^{-2\lambda} = 1 - 2M/r. \tag{4.36}$$

The isotropic form of the solution was given by Eddington [29], and the line-element takes the form (**??**) with

$$e^{\nu} = \frac{2r - M}{2r + M}, \quad e^{\mu} = \left(1 + \frac{M}{2r}\right)^2. \tag{4.37}$$

The exterior Schwarzschild solution has also been given in various coordinate systems by Finkelstein [34], and the maximal analytic extension was provided by Kruskal [52]. Although over 70 years old, forms of the Schwarzschild exterior solution are still a source of research, as shown by [], who gave a highly symmetric "orthogonal form" for the spacetime.

In the same year Schwarzschild published [85] his famous interior solution, describing a constant density fluid sphere. For a constant density, $\rho = \rho_0$, the solution (in Schwarzschild coordinates) is given by

$$e^{\nu} = \frac{3}{2}b - \frac{1}{2}\sqrt{1 - Br^2}, \quad e^{-2\lambda} = 1 - \frac{1}{3}\rho_0 r^2, \tag{4.38}$$

where b and B are constants. The pressure assumes the analytic form

$$p = \rho_0 \frac{b - \sqrt{1 - Br^2}}{\sqrt{1 - Br^2} - 3b}.$$

Matching to (4.36) provides the relations

$$b = (e^{\nu})_s = \sqrt{1 - 2M/r_s},$$

$$B = \frac{\rho_0}{3} = \frac{2M}{r_s^3} = \frac{(1 - b^2)^3}{4M^2}.$$

Although the solution must be one of the most famous SSSPF solutions known, it is easily seen to be unphysical. For, since ρ is constant and $p = p(r)$, the sound speed is infinite and (3.26) is violated. Later, Buchdahl and Land [13] were to generalise the Schwarzschild interior solution to what they claimed was a "relativistic incompressible fluid sphere".

In a flurry of papers in 1939, work on SSSPF solutions reappeared with three seminal papers by Volkoff [91], Tolman [89] and Oppenheimer and Volkoff [80]. That the last two of these articles appear back to back in the same issue of the Journal of Mathematical Physics is no coincidence; on reading them they clearly influenced one another, thought they approach the problem of fluid spheres from two completely different viewpoints.

In the works involving Volkoff, the main thrust comes from physical considerations, with the derivation of the equation of state of Fermi particles given in [80], and the emphasis is on numerical integration of the field equations, including the famous Oppenheimer-Volkoff equation (??). Similarly, in [91], though Volkoff proposes a generalisation of the form of g_{11} in (4.38), proposing

$$e^{-2\lambda} = 1 - \frac{1}{3}\rho_0 r^2 + \frac{A}{r}, \tag{4.39}$$

with A constant, the question of a closed from for g_{00} is not pursued, and is left to be cleared up by Wyman some ten years later [99].

Tolman's work approaches the SSSPF field equations for an almost exclusively mathematical point of view by assuming various ad hoc forms (ansätze) for the metric functions in Schwarzschild coordinates. He shows that by assuming the metric coefficient forms

$$e^{\nu} = a, \quad e^{\nu+\lambda} = b, \quad e^{-2\lambda} = 1 - cr^2 \tag{4.40}$$

for constant a, b and c the solutions obtained are the Einstein universe [], the Schwarzschild-deSitter (or Köttler) solution, and the Schwarzschild interior solutions respectively.

Tolman makes four other proposals for the metric coefficients, all of which produce original solutions. Many of Tolman's solutions (or subsets thereof) have been rediscovered, with authors apparently all too unaware of this key work in the field.

Traditionally, Tolman's solutions are denoted by a Roman numeral — and the first of Tolman's original solutions is the Tolman IV solution, where starting from the ansatz

$$e^{2\nu} = b(1 + ar^2),$$

with a and b constants, Tolman solved for

$$e^{2\lambda} = \frac{1 + 2ar^2}{(1 - ar^2)(1 - cr^2)},$$

where c is a further constant, producing analytic forms for $p(r)$ and $\rho(r)$:

$$p = \frac{a - c - 3acr^2}{1 + 2ar^2}, \quad \rho = \frac{a + 3c + 3acr^2}{1 + 2ar^2} + \frac{2a(1 - cr^2)}{(1 + 2ar^2)^2},$$

and even a closed form equation of state whcih takes the form of a quadratic in p for ρ

$$\rho = \rho_0 - 5(p_0 - p) + 8\frac{(p_0 - p)^2}{p_0 + \rho_0}. \tag{4.41}$$

Although, as we have said, apparently principally motivated form a mathematical point of view, Tolman was sufficiently interested in the physical properties of his solutions to investigate, where reasonable, whether certain reality conditions held. He showed that (??) satsifies (3.22)–(3.26) provided

$$0 < \frac{2a =}{23} < c < a; \quad \rho_0 \geq 3p_0 \tag{4.42}$$

and that, given these restrictions on the constants, (??) was also guaranteed to hold. Having showed that a, b and c can be expressed in terms of either of the pairs $\{p_0, \rho_0\}$ or $\{M, r_s\}$, Tolman in effect suceeded in producing, given (4.42), the first SSSPF solution which could be said to be regular, realistic and simple.

For his next solution, Tolman investigated what happens when e^ν is a simple power-law and found

$$e^\nu = Br^n, \quad e^{-2\lambda} = ar^m + \frac{1}{1+2n-n^2}, \quad m \equiv \frac{2(1+2n-n^2)}{1+n}, \tag{4.43}$$

where a, B and n are constants. Tolman studies in detail the case with $n = \frac{1}{2}$, since then the central conditions reduce to a radiation fluid, with $\rho_0 = 3p_0$. Despite this nice characteristic, Tolman shows (as does Lieibovitz 30 years later [71]) that this solution is irregular. In fact the solution is irregular for general n, but with $n = 0$ we recover to the Einstein static universe. en selecting $a = 0$, the solution reduces to the well-known isothermal (or gamma-law) solution usually known by the names of its rediscoverers Misner and Zapolsky [75]. Tolman perhaps overlooked this important subclass because it would have required letting his constant $R \to \infty$. Whatever the reason for the omission, in the notation of (4.43), the solution has equation of state

$$p = \frac{n}{2-n}\rho.$$

We do not give the general forms for p and ρ, since they are quite complicated, but they are detailed in [89].

For Tolman's third new solution (the Tolman VI solution), the assumption is that g_{11} is constant leading to

$$e^\nu = Ar^{1-n} + Br^{1+n}, \quad e^{2\lambda} = 2 - n^2, \tag{4.44}$$

with A, B and n constants. The density and pressure have the forms

$$\rho = \frac{1-n^2}{(2-n^2)r^2}, \quad p = \frac{A(1-n)^2 - B(1+n)^2 r^{2n}}{(2-n^2)\left(A - Br^{2n}\right) r^2}. \tag{4.45}$$

To preserve the signature of the metric, n must be restricted to lie in the interval $[0, \sqrt{2})$. The form for ρ clearly implies an irregular solution. Once again, Tolman seems not to have noticed that the Misner-Zapolsky solution for an isothermal equation of state is contained within one of his solutions, though this time his omission is rather more puzzling, since, as pointed out by Collins [18], one need only set A or B to zero, giving the equation of state

$$p = \frac{1 \mp n}{1 \pm n}\rho \tag{4.46}$$

depending on which constant is set to zero. However, Tolman does note that the choice $n = \frac{1}{2}$ produces a simple solution, which is further discussed by Kuchowicz [56] and generalised by Wyman [99].

The Tolman VII solution arises from an assumption that $e^{-2\lambda}$ is polynomial in r, of the form

$$e^{-2\lambda} = 1 - Ar^2 + B^2 r^4, \to e^\nu = C \sin\left\{\log\left[D\left(e^{-\lambda} + Br^2 - \frac{A}{2B}\right)^{\frac{1}{2}}\right]\right\} \tag{4.47}$$

wth A, B, C and D arbitrary constants. Although the density has a simple form, the pressure is rather complicated, their forms being

$$\rho = 3A - 5B^2 r^2, \quad p = -A + B^2 r^2 + 2be^{-\lambda}\left(Ce^{-2\nu} - 1\right)^{\frac{1}{2}} \tag{4.48}$$

respectively. Tolman discarded this solution as being too complicated, but with hindsight we can use the results of (3.29) to establish immediately that at least has the virtue that ρ_0 and p_0 are finite. That the former is true is obvous from (4.48), but the latter takes a bit of work to prove from these equations. In any case other people have found the solution suffieciently interesting to investigate it further, notably Durgapal and Rawat [28], while Rawat [82] claims the solution is unstable under radial perturbations. Bayin [?] uses the same ansatz for g_{11}, but with different restrictions on A and B, and produces a different (but unrealistic) solution.

As his final solution, Tolman presents the irregular solution defined by

$$e^{\nu+\lambda} = Br^a, \quad e^{-2\lambda} = \frac{2}{bd} - \left(\frac{2C}{r}\right)^d - \left(\frac{r}{r_s}\right)^b \tag{4.49}$$

where the constants a, b and d may be related parametrically through a subsidiary constant A by

$$a \equiv \frac{(2-A)(1+A)}{A-3}, \quad b \equiv \frac{2(A^2-2A-1)}{A-3}, \quad d \equiv \frac{7-2A-A^2}{A-3}. \tag{4.50}$$

Tolman points out that this may be regarded as a generalisation of the Köttler solution, which is recovered by setting $A = 2$, giving $a = 0$, $b = 2$ and $d = 1$.

Following Tolman's work in Schwarzschild coordinates, the first article which followed a simliar approach in *isotropic* coordinates seem to have been by Narlikar et al [?], who studied two ansätze related to the forms of g_{11} in (**??**). Each of the ansätze divides into three sub-cases, depending on the values of constants involved. The first assumption used is very similar to the anstaz used for g_{00} in the Tolmman IV solution (**??**):

$$e^{-2\mu} = Ar^{1+k} + Br^{1-k}, \tag{4.51}$$

and for all sub-case leads to the density distribution

$$\rho = (1-k^2)\left(A^2 r^{2k} + B^2 r^{-2k}\right) + 2AB(1+5k^2). \tag{4.52}$$

The subcases, however, give different forms of g_{00}, depending on the value of k. They may be stated as

$$\frac{1}{\sqrt{2}} < k \leq 1 \quad , e^{\nu} = e^{\mu}\left(C_1 r^{1-a} + C_2 r^{1+a}\right), \quad a \equiv \sqrt{2k^2-1}, \tag{4.53}$$

$$k = \frac{1}{\sqrt{2}}, \quad e^{\nu} = re^{2\mu}(C_1 \ln r + C_2), \tag{4.54}$$

$$0 \leq k < \frac{1}{\sqrt{2}}, \quad e^{\nu} = ?. \tag{4.55}$$

As the authors point out, when $k = 1$ in solution (4.53) has constant ρ and is the interior Schwarzschild solution, which this set of solutions may thus be regarded as generalising.

The authors' second ansatz is $\mu' = Ar^{-1}$, giving three subclasses which depend on the sign of $2A^2 + 4A + 1$. Parts of this paper were later summarised by Kuchowicz [66].

Kuchowicz (1967a) [53]

Like Wyman [99], Kuchowicz generalises the Tolman VI solution (??), but has n imaginary. Defining $u \equiv r/r_S$, the solutions may be stated as:

Ku1

$$
\begin{aligned}
e^{\nu} &= u\left[a\cos(n\ln u) + b\sin(n\ln u)\right] \\
e^{-2\lambda} &= \frac{1}{2+n^2} + Cu^{2\alpha}\left[(2a+nb)\cos(n\ln u) + (2b-na)\sin(n\ln u)\right]^{-\alpha} \\
\alpha &\equiv \frac{2(n^2+2)}{n^2+4} \\
a &= sqrt1-\delta \\
b &= \frac{3\delta-2}{2m\sqrt{1-\delta}} \\
C &= \frac{2\sqrt{1-\delta}}{2-\delta}\left(\frac{n^2+1}{n^2+2}-\delta\right)^{\alpha} \\
\rho &= \frac{1}{r^2}\left\{\frac{1+n^2}{2+n^2} - Cu^{2\alpha}\frac{\left[(6+2n^2)a+nb\right]\cos(n\ln u) + \left[(6+2n^2)b-na\right]\sin(n\ln u)}{\left[(2a+nb)\cos(n\ln u)+(2b-na)\sin(n\ln u)\right]^{1+\alpha}}\right\}
\end{aligned} \tag{4.56}
$$

The form of ρ given in the original contains a spurious constant. Kuchowicz notes that singularities may result in the metric coefficient g_{11}, and in the density if $C \neq 0$. For $C = 0$ the solution simplifies to:

Ku1a

$$
\begin{aligned}
e^{-2\lambda} &= \frac{1}{n^2+2} \\
\rho &= \frac{n^2+1}{(n^2+2)r^2}
\end{aligned} \tag{4.57}
$$

These solutions (and the general solution) are irregular at the centre. In addition, Kuchowicz later shows [56] that for the case of vanishing C, $p < 0$ somewhere within the sphere.

Kuchowicz (1967b) [54]

This paper summarises a solution also given later in [56]. Kuchowicz regards it as a generalisation of the interior Schwarzschild solution. It may be stated as:

Ku2

$$
\begin{aligned}
e^{-2\lambda} &= a + br^2 \\
\rho &= -3b + (1-a)r^{-2} \\
0 \leq\ & a < 1
\end{aligned} \tag{4.58}
$$

The form of e^ν is dependent on the value of a, and divides into three subcases, *viz*

Ku2a

$$
\begin{aligned}
\frac{1}{2} &< a < 1 \\
e^\nu &= r^{1-\alpha}\left[C_1(1+v)^\alpha + C_2(1-v)^\alpha\right] \\
v &\equiv \sqrt{1+\frac{br^2}{a}} \\
\alpha &\equiv \sqrt{2-a^{-1}}
\end{aligned}
\tag{4.59}
$$

Ku2b

$$
\begin{aligned}
a &= \frac{1}{2} \\
e^\nu &= r\left(C_1 + C_2 \ln\left|\frac{1+v}{1-v}\right|\right) \\
v &\equiv \sqrt{1+2br^2}
\end{aligned}
\tag{4.60}
$$

Ku2c

$$
\begin{aligned}
0 &< a < \frac{1}{2} \\
e^\nu &= r(C_1 \sin w + C_2 \cos w) \\
w &\equiv \kappa \ln\left(\frac{1+\sqrt{1+\beta r^2}}{r}\right)
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
\kappa &\equiv \sqrt{\frac{1-2a}{a}} \\
\beta &\equiv \frac{b}{a}
\end{aligned}
\tag{4.62}
$$

Kuchowicz also gives the form for the pressure for (4.59). Since $a \neq 1$, none of the solutions are physically regular at the centre.

Kuchowicz (1967c) [55]

This is intended as a review paper, but in fact presents nine new solutions. They are all listed below in the reviews of [56], [57].

Durgapal and Gehlot (1968) [24]

This paper presents a rediscovery of the Tolman V solution (??) without acknowledging the original work. The authors state they have solved Einstein's equations for "all spherically symmetric static massive bodies of extremely high density". The solutions have, in fact, infinite central density, and are thus unrealistic. Furthermore, the Tolman V solutions are only a few of the many irregular SSSPF solutions to the SSSPF field equations!

Kuchowicz (1968a) [56]

In this paper, Kuchowicz investigates conditions under which equations (??)–(??) can be solved, by defining

$$u(r) \equiv e^{\nu} \tag{4.63}$$

and obtaining

$$e^{-2\lambda}u'' - \left(\frac{e^{-\lambda}\lambda'}{2} + \frac{e^{-\lambda}}{r}\right)u' + \frac{1}{2r^2}\left(2 - e^{-\lambda}\lambda' r - 2e^{-\lambda}\right)u = 0 \tag{4.64}$$

Regarding this as a special case of the general linear equation

$$f_1 u'' + f_2 u' + f_3 u = 0, \tag{4.65}$$

Kuchowicz's first three assumptions are of proportionality between the pairs of functions $\{f_1, f_2\}$,$\{f_2, f_3\}$, and $\{f_1, f_3\}$ respectively.

Kuchowicz claims that the most fruitful form of the proportionality between f_1 and f_2 is to have

$$f_1 = \frac{bf_2}{r} \tag{4.66}$$

where b is a constant, and deals with the cases $b = 1$, $b \neq 1$ separately.

When $b = 1$, Kuchowicz supplies three distinct solutions. The first is equivalent to Tolman VI (??), the second is a degenerate case of the first, and results in

$$\begin{aligned} e^{\nu} &= Ar + Br\ln r \\ e^{-2\lambda} &= \frac{1}{2} \\ \rho &= \frac{1}{2r^2} \\ p &= \frac{1}{2r^2}\left(1 + \frac{2b}{a + b\ln r}\right) \end{aligned} \tag{4.67}$$

The third solution had already been given by Kuchowicz as (4.56). It is clear that solution (4.67) is irregular at the centre. Kuchowicz mistakenly says that the Tolman VI solution also has this property, not noticing that in the special case where

$$e^{\nu} = A + Br^2 \tag{4.68}$$

the central density and pressure actually vanish.

When $b \neq 1$, the assumption reduces to

$$e^{-2\lambda} = Cr^{2(1-b)} \tag{4.69}$$

Kuchowicz shows that the resulting equation for y is reducible, by substitution, to a Bessel equation, with solution

$$\begin{aligned} e^{2\nu} &= r^{1+b}\left[AJ_\nu(x) + BJ_{-\nu}(x)\right] \\ x &\equiv \frac{r^{b-1}}{(b-1)\sqrt{C}} \\ \nu &\equiv \frac{\sqrt{(1+b)^2+4b}}{2(b-1)} \end{aligned} \tag{4.70}$$

with the Bessel function Y_ν replacing J_ν for integer values of ν. Again, A and B are constants.

Kuchowicz points out that, for $\nu = n + \frac{1}{2}$, where n is a non-negative integer, the form of $e^{2\nu}$ is elementary. This is equivalent to demanding that b be given by

$$b = \frac{n^2+2n+1 \pm \sqrt{2n^2+2n+1}}{n(n+1)} \tag{4.71}$$

Kuchowicz provides the examples for

$$\begin{aligned} &\nu = \frac{1}{2} \qquad b = 0 \\ e^{-2\lambda} &= Cr^2 \\ e^\nu &= r * \left[A * \sin\left(-\frac{1}{r\sqrt{C}}\right) + B * \cos\left(-\frac{1}{r\sqrt{C}}\right)\right] \\ \rho &= \frac{1}{r^2} - 3C \end{aligned} \tag{4.72}$$

$$\begin{aligned} &\nu = \frac{3}{2} \quad b = \frac{3\pm\sqrt{5}}{2} \\ e^{-2\lambda} &= Cr^{-1\mp\sqrt{5}} \\ e^\nu &= r\left[A\left(\frac{\sin z}{z} - \cos z\right) + B\left(\frac{\cos z}{z} + \sin z\right)\right] \\ z &\equiv \frac{2r^{(1\pm\sqrt{5})/2}}{(1\pm\sqrt{5})\sqrt{C}} \\ \rho &= \frac{1}{r^2} \pm C\sqrt{5}r^{-3\mp\sqrt{5}} \end{aligned} \tag{4.73}$$

Despite the abundance of solutions obtained by this method, we can easily detect from the form of $e^{2\lambda}$ that all these solutions will be irregular.

Kuchowicz then shows that the choice $f = 0$ leads to the interior Schwarzschild solution (??), while $f \propto r^{-1}$ produces:

$$e^{-2\lambda} = a + br^2$$

$$\rho = -3b + \frac{1-a}{r^2} \tag{4.74}$$

By defining new constants c and K such that

$$a = 1 - \frac{c}{K}$$

$$b = \left(\frac{1}{K} - 1\right)\frac{a}{r_S^2} \tag{4.75}$$

Kuchowicz finds that the solutions fall into seven categories, depending on the values of c and K. The three solutions for $K = 1$ (*ie* $b = 0$) have already been given at the start of this section. When $K \geq \frac{2}{3}$ with $K \neq 1$, there are again three subcases for $K > 2c$, $K = 2c$ and $c < K < 2c$. The solutions are lengthy and are not presented here. Finally, the assumption $K = c$, produces the solution (4.70).

The next solution presented follows from

$$e^{-2\lambda} = -1 + \frac{2b}{n-2} r^n \tag{4.76}$$

which results in e^ν being given in terms of hypergeometric functions.

Another solution for e^ν in terms of hypergoemetric functions follows from

$$e^{-2\lambda} = a - 2\ln r \tag{4.77}$$

Finally, Kuchowicz examines solutions resulting from

$$e^{-2\lambda} = a + br^{2/a} \tag{4.78}$$

and produces another hypergeometric equation for e^ν. Unfortunately, the coefficient of y' in the original equation — Kuchowicz's (7.4) — is incorrect. In our notation, the correct equation is:

$$r^2\left(a + br^{2/a}\right)(e^\nu)'' + r\left[\left(\frac{1}{a} - 1\right)br^{2/a} + a\right]e^\nu + \left(\frac{1}{a} - 1\right)\left(a + br^{2/a}\right) = 0 \tag{4.79}$$

The correct solution for e^ν then comes from this hypergeometric equation. Since the three subsequent solutions Kuchowicz gives ($a = \frac{3}{2}$, $a = 1$ and $a = -\frac{1}{2}$ follow from his version of (4.79), they are also incorrect.

The final section of the paper deals with ansätze for e^ν, and gives no new explicit solutions.

Buchdahl and Land (1968) [13]

examine the equation of state

$$p = \rho - \rho_S \tag{4.80}$$

which they claim represents relativistic incompressibility. They find one exact solution for (4.80) which is unphysical near the centre. They then resort to numerical integration to find physically realistic solutions.

Kuchowicz (1968b) [57]

This paper covers five separate ansätze for $\nu(r)$. The first ansatz has already been considered as (4.87) and results in a non-elementary solution. The second ansatz considered is Tolman's ansatz for his solution (**??**), which goes unacknowledged. However a new solution is produced for the special case

$$\begin{aligned} e^{2\nu} &= ar^{2(1+\sqrt{2})} \\ e^{-2\lambda} &= C + (\sqrt{2}-2)\ln r \\ \rho &= \frac{(sqrt2-2)\ln r + \sqrt{2} - 3 + C}{r^2} \\ p &= \frac{(2\sqrt{2}+3)C - 1 - (sqrt2+2)\ln r}{r^2} \end{aligned} \tag{4.81}$$

Kuchowicz determines the boundary conditions, which further simplify the forms of p and ρ.

The third ansatz used results in a non-elementary solution for which we only give the metric coefficients:

$$\begin{aligned} e^{2\nu} &= ae^{br^2/2} \\ e^{-2\lambda} &= 1 + r^2 e^{-br^2/2}\left[C - \frac{b}{2e}\mathrm{Ei}\left(1+\frac{1}{2}br^2\right)\right] \end{aligned} \tag{4.82}$$

The pressure and density similarly depend on exponential integrals.

Kuchowicz then proceeds to state that there exist two classes of elementary solutions which follow from the ansatz

$$\nu' = ar^n \tag{4.83}$$

provided that n is given by

$$n = -1 \pm \frac{2}{z} \tag{4.84}$$

where z is a natural number. An example is given for $n = -3$

The next solution:

$$\begin{aligned} e^{\nu} &= a(r+b) \\ e^{-2\lambda} &= (c - \frac{4}{b^2})r^2 - \frac{2r}{b} + 1 + \frac{4r^2}{b^2}\ln\left|1 + \frac{b}{2r}\right| \end{aligned} \tag{4.85}$$

is simple but irregular. It is rediscovered by Bayin [**?**].

Finally, the assumption

$$\begin{aligned} e^{\nu} &= ar^{1\pm\sqrt{2}} \\ e^{-2\lambda} &= c - \frac{2}{2\pm\sqrt{2}}\ln r \end{aligned} \tag{4.86}$$

The solution is physically irregular at the centre. A more general solution is presented by using the above form for $e^{-2\lambda}$, but the solution requires a quadrature.

Kuchowicz (1968c) [58]

In this paper, the author gives a very thorough discussion of the conditions for physical regularity described in §??. He also discusses the boundary conditions for joining onto the exterior Schwarzschild solution, and for joining two interior solutions. Though no solutions are explicitly given, the two ansatze

$$e^{2/nu} = ae^{br} \tag{4.87}$$

$$e^{-2\lambda} = a + br \tag{4.88}$$

are investigated, and it is shown that the first results in a non-elementary quadrature, while the second leads to a hypergeometric equation for e^{ν}.

Kuchowicz (1968d) [59]

Here, Kuchowicz presents what he terms "extensions of the external Schwarzschild solution" by generalising the solutions found by assuming the forms of $e^{2\nu}$ and $e^{2\lambda}$ for solution (??).

The solutions are:

$$\begin{aligned} e^{-2\lambda} &= 1+\frac{a}{r} \\ e^{\nu} &= Au + B\left\{\frac{r^2}{2a^2} - \frac{5r}{4a} - \frac{15}{4} + \frac{15}{8}u\ln\left[1+\frac{2r(1+u)}{a}\right]\right\} \\ \rho &= 0 \\ p &= \frac{2B}{a^2}e^{-\nu} \\ u &\equiv \sqrt{1+\frac{a}{r}} \end{aligned} \tag{4.89}$$

which reduces to the exterior Schwarzschild solution for $B = 0$. The solution for $A = 0$ gives a special case of Tolman's solution (??) with $n = 1$, which Kuchowicz does not acknowledge (the form of the pressure given by Kuchowicz for this particular solution is also too small by a factor of two). Except for the Schwarschild exterior solution, the solution is physically irregular at the centre. The second solution given by Kuchowicz reads:

$$\begin{aligned} e^{2\nu} &= 1+\frac{a}{r} \\ e^{-2\lambda} &= \left(1+\frac{a}{r}\right)\left[1+2b(a+2r)^2\right] \\ p &= \frac{2b(a+2r)^2}{r^2} \\ \rho &= -\frac{b(a+2r)(5a+6r)}{r^2} \end{aligned} \tag{4.90}$$

The forms for $e^{-2\lambda}$ and p are given incorrectly in the original.

The exterior Schwarzschild solution is recovered for $b = 0$. Otherwise, the solution is invalid throughout the sphere, being physically irregular at the centre.

Kuchowicz also mentions that the other obvious generalisation of the exterior Schwarzschild solution, *viz* $e^{\nu} = e^{-\lambda}$ leads to Kottler's solution (**??**), though the form Kuchowicz gives for the metric coefficients is incorrect.

Whittaker (1968) [97]

Whittaker presents a solution with

$$\rho + 3p = K, \tag{4.91}$$

with K a constant. He claims that such a solution is the analogy of the Schwarzschild interior solution for constant "gravitational mass density". The solution is given in terms of Schwarzschild coordinates as

$$\begin{aligned} e^{2\nu} &= n\left(1 + B - \frac{B\sqrt{1-\alpha^2 r^2}}{\alpha r}\arcsin \alpha r\right) \\ e^{-2\lambda} &= (1+B)(1-\alpha^2 r^2) - \frac{B}{\alpha r}(1-\alpha^2 r^2)^{3/2}\arcsin(\alpha r) \\ \alpha^2 &\equiv \frac{1}{2}ln \\ B &\equiv \frac{K}{2\alpha^2} \end{aligned} \tag{4.92}$$

where l and n are constants. A more general form for $e^{2\nu}$ is presented, but Whittaker concludes that the additional constant involved must vanish for the metric coefficients to remain finite at $r = 0$. It is easily seen, however, that

$$\frac{dp}{d\rho} = -\frac{1}{3} \tag{4.93}$$

and so the solution is unrealistic.

Durgapal and Gehlot (1969) [?]

The authors present a composite stellar model, matching the interior Schwarzschild solution to the Tolman V solution. This avoids the problem of an infinite central density, but the sound speed in the interior Schwarzschild region is infinite, and so unrealistic.

Heintzmann (1969) [41]

Heintzmann gives seven solutions in Schwarzschild coordinates, two of which come from a generating technique, the other 5 arising from the ansatz

$$e^{\nu} = a + br^{2n} \tag{4.94}$$

for various values of the constant n. Many of the solutions are given incorrectly, and the corrected versions are given later. But first we describe the generating technique, which utilises the same formalism as Buchdahl [10].

If a particular solution, e^{ν}, w_0 of (**??**) is known, and w_0 does not contain a free constant present in ν, then

$$w = w_0 + (1 + 2q\nu'(q))e^{-2\nu}\exp\left[4\int\frac{dq}{1+2q\nu'}\right] \tag{4.95}$$

is a new solution with one more constant. Thus an $n+1$-parameter solution is generated from an n-parameter one.

Heintzmann applies his algorithm to the interior Schwarzschild solution (??) and generates the following solution

$$
\begin{aligned}
e^{\nu} &= a - bu \\
e^{-2\lambda} &= u^2(1 - c\delta r^2 v) \\
u &\equiv 1 - \delta r^2 \\
v &\equiv (u-\beta)^{2\alpha/(\beta-\alpha)} + (u-\alpha)^{2\beta/(\alpha-\beta)} \\
\rho &= 3\delta + c\delta u^2 v \left[3 + \frac{\delta r^2}{u(u-\alpha)(u-\beta)}\right] \\
p &= \frac{\delta(3bu-a)}{a-bu} - c\delta u^2 v \left[1 + \frac{2b\delta r^2}{u(a-bu)}\right] \\
\alpha &\equiv \frac{1}{4}\left(\frac{a}{b} + \sqrt{8 + \frac{a^2}{b^2}}\right) \\
\beta &\equiv \frac{1}{4}\left(\frac{a}{b} - \sqrt{8 + \frac{a^2}{b^2}}\right)
\end{aligned}
\tag{4.96}
$$

where a, b, c and δ can be given in terms of r_S and the free parameter u_S as:

$$
\begin{aligned}
a &= \frac{(3\alpha-2)u_S^2 + 2(1-\alpha)}{2\sqrt{1-\alpha}(1-u_S^2)} \\
b &= \frac{\alpha u_S}{2\sqrt{1-\alpha}(1-u_s^2)} \\
c &= \frac{\alpha - 1 + u_S^2}{u_S^2(1-u_S^2)v_S} \\
\delta &= \frac{1-u_S^2}{r_S^2}
\end{aligned}
\tag{4.97}
$$

Heintzmann notes that if $\delta > 0$ then:

$$
\begin{gathered}
p'(r) < 0 \quad \text{and} \quad \rho(r)' < 0 \\
0 \leq \quad \alpha \quad \leq \frac{8}{9}
\end{gathered}
\tag{4.98}
$$

whereas, for $\delta = 0$ the solution is flat space, and for $\delta < 0$, ρ' may be positive near the surface. The behaviour of $dp/d\rho$ is not mentioned, and so the solution has not been shown to be realistic, but since the solution reduces to the realistic Tolman IV solution (??) when $a = 0$, it is likely that (4.96) is also realistic in some neighbourhood of $a = 0$.

Two further solutions are generated from Buchdahl's solution (??). By taking A and B to vanish separately, the solutions are:

$$\begin{aligned} e^{\nu} &= B(5 + cr^2)\sqrt{2 - cr^2} \\ e^{2\lambda} &= \frac{2(1 + cr^2)}{2 - cr^2 - 3c\delta r^2(5 - 4cr^2)} \end{aligned} \tag{4.99}$$

and

$$\begin{aligned} e^{\nu} &= A(1 + cr^2)^{3/2} \\ e^{2\lambda} &= \frac{2(1 + cr^2)}{2 - 3c\delta r^2(1 + 4cr^2)^{-1/2}} \end{aligned} \tag{4.100}$$

respectively, with A, B, c and δ constants.

The solutions using ansatz (4.94) come from choosing $n \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}\}$. The solution with $n = \frac{1}{2}$ is Kuchowicz's solution (4.85) (unacknowledged), which is later rediscovered by Bayin [?]. Choosing $n = 1$ gives Wyman's solution (??) (also unacknowledged).

From §??, we know that the form of (4.94) only admits physically regular solutions at the centre for $n = 1$, and so Wyman's solution is the only physically realistic one.

Kuchowicz (1969) [60]

This paper simply summarises results available in [63].

Leibovitz (1969) [71]

This paper is mainly concerned with general theorems, as in §??, in Schwarzschild coordinates. An example is that if (??) and (??) are to hold, and $\rho \propto r^n$ then $n = -2$. Leibovitz also re-derives (??), but gives $p(r)$ incorrectly.

Krori (1970) [48]

The author rediscovers the Tolman V solution (??), and gives a composite stellar model by matching to the Schwarzschild exterior and interior solutions. No new solutions are presented.

Kuchowicz (1970) [61]

Three ansätze are used in this paper for $e^{2\nu}$ in Schwarzschild coordinates. The first to be used is

$$\begin{aligned} e^{\nu} &= C_1 r^{\alpha} + C_2 r^{\beta} \\ \alpha + \beta &= D + 2 \\ \alpha\beta &= -D - 1 \end{aligned} \tag{4.101}$$

Kuchowicz deduces values for D for which the solution is elementary. The first of these is

$$
\begin{aligned}
D &= \frac{2(2+\sqrt{34})}{15} \\
\alpha &= \frac{5+\sqrt{34}}{3} \\
\beta &= \frac{3-\sqrt{34}}{3} \\
e^{-2\lambda} &= -\frac{1}{(1+\alpha D)} + Cr^{-2D} - F(r) \qquad (4.102)
\end{aligned}
$$

for which there is a trichotomy of solutions depending on the sign of C_1C_2:

$$
F(r) = \frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}r^{2D}}\sqrt{\frac{C_2}{C_1}}\arctan\left[\sqrt{\frac{(1+\alpha)C_1}{(1+\beta)C_2}}r^{2D}\right] \qquad (4.103)
$$

$$
F(r) = 0 \qquad (4.104)
$$

$$
F(r) = \frac{4}{(1+\alpha)^{3/2}(1+\beta)^{1/2}r^{2D}}\sqrt{-\frac{C_2}{C_1}}\ln\frac{(1+\beta)C_2-(1+\alpha)C_1r^{4D}+2\sqrt{-C_1C_2(1+\alpha)(1+\beta)}r^{2D}}{(1+\beta)C_2+(1+\alpha)C_1r^{4D}} \qquad (4.105)
$$

for $C_1C_2 > 0$, $C_1C_2 = 0$ and $C_1C_2 < 0$ respectively. Kuchowicz gives the form for p and ρ for the case $F(r) = 0$, but not for the others. This solution is a special case of (**??**). The second value of D for which Kuchowicz is able to derive elementary exact solutions is

$$
\begin{aligned}
D &= \frac{4+2\sqrt{74})}{35} \\
\alpha &= \frac{7+\sqrt{74}}{5} \\
\beta &= \frac{5-\sqrt{74}}{7} \qquad (4.106)
\end{aligned}
$$

This time there are only two solutions, for $C_2 = 0$ and $C_2 \neq 0$. Again, the case $C_2 = 0$ is contained within (**??**). The case $C_2 \neq 0$ is extremely complicated, and is not shown here.

Kuchowicz then considers the case

$$
2D = -n\sqrt{D^2+8D+8} \qquad (4.107)
$$

and shows that this class of solutions contains Wyman's solution (**??**) when $n = D = 0$, while choosing $n = 2$ recovers Kuchowicz's earlier solution (**??**).

The other two ansätze investigated are

$$
e^{\nu} = (a + b\ln r)r^n \qquad (4.108)
$$

$$
e^{\nu} = are^{br} \qquad (4.109)
$$

but neither give simple solutions.

Krori (1971) [49]

As with his previous paper [48], this paper deals with composite stars constructed from various Tolman solutions by assuming a core described by the Tolman IV solution, and matching this to an outer region described by the Tolman V/VI solutions.

Kuchowicz (1971a) [62]

First, equation (??) is used with the ansatz

$$\begin{aligned} \mu''(z) + (\mu')^2 + e^{-4\mu} &= \left[a(ae^{4\mu})^\alpha - b(b-1)\right](ae^{4\mu})^\beta \\ \alpha &\equiv \frac{c}{2-c} \\ \beta &\equiv \frac{2}{2-c} \\ c &\equiv \frac{2(1-2b)}{1+2n} \end{aligned} \tag{4.110}$$

with n an integer, but the form of $p(r)$ is not simple. Choosing $c = 0$ gives (??), which is equivalent to the Tolman VI solution (??).

A solution in isotropic coordinates is also given:

$$\begin{aligned} e^{2\mu} &= ae^{br^2} \\ e^{\nu} &= c\exp\left[\frac{(1+\sqrt{2})br^2}{2}\right] + d\exp\left[\frac{(1-\sqrt{2})br^2}{2}\right] \\ \rho &= -\frac{6b + b^2r^2}{a}e^{-br^2} \end{aligned} \tag{4.111}$$

but (??) is violated near the centre.

Kuchowicz (1971b) [63]

summarises parts of [66].

Kuchowicz (1972a) [65]

summarises parts of [68].

Kuchowicz (1972b) [66]

This paper puts forward no fewer than seventeen ansätze for solutions in isotropic coordinates, but none are shown to be realistic. First, six forms for e^μ are proposed:

$$e^{2\mu} = ae^{br}$$

$$\rho = -\frac{b(ar+8)}{4are^{br}} \tag{4.112}$$

results in μ and p not elementary.

The second ansatz is (4.111), the third is (**??**). Kuchowicz claims that his fourth ansatz

$$e^{2\mu} = ae^{br^2} \tag{4.113}$$

leads to non-elementary functions for μ and p, but Bayin [**?**] investigates it and shows otherwise (4.135). The fifth ansatz is (**??**) with $c = 2$, giving (**??**) in isotropic coordinates. The sixth is:

$$e^{2\mu} = a\exp\left[\frac{(2-\sqrt{2})r^2}{4r_s^2} + b\exp\left(\frac{r}{r_S}\right)^2\right] \tag{4.114}$$

but again ν is not elementary.

For the remaining 11 solutions, Kuchowicz writes (**??**) is a particularly simple form in isotropic coordinates by defining

$$\begin{aligned} u'(q) &= \frac{1}{2}u^2 - v^2 \\ u &\equiv 2\left[x'(q) - \frac{y'(q)}{y}\right] \\ v &\equiv 2x'(q) \\ q &\equiv r^2 \end{aligned} \tag{4.115}$$

The ansätze considered are

$$v = 0 \tag{4.116}$$

but (**??**) and (**??**) are unfulfilled.

$$v = a \tag{4.117}$$

which gives a very complicated form for p.

$$v = \frac{a}{r^2} \tag{4.118}$$

which does not satisfy (**??**) and (**??**).

$$v = ar^n, n \neq -2 \tag{4.119}$$

give very complicated formulae, for which only ν is given explicitly.

$$\frac{v^2}{u} = ar^2 (a \neq 1) \tag{4.120}$$

$$v^2 overu = \frac{1}{r^2} \tag{4.121}$$

$$v^2 overu = a \tag{4.122}$$

all give very complicated expressions for $\rho(r)$ and $p(r)$.

The next solution

$$
\begin{aligned}
\frac{v}{u} &= a \\
e^{2\nu} &= c\left|\frac{ar^2}{\alpha}+b\right|^{\alpha} \\
e^{2\mu} &= d(e^{2\nu})^{\beta} \\
\alpha &\equiv \frac{2a}{2a^2-1} \\
\beta &= \frac{\alpha(1-a)}{a}
\end{aligned}
\tag{4.123}
$$

is fairly simple, but is not investigated further by Kuchowicz, although Bayin [?] does.

The assumption

$$
v^2 = au + b \tag{4.124}
$$

leads to three solutions:

$$
\alpha^2 \equiv a^2 + 2b > 0 \quad \text{gives} \quad u = \alpha \tanh(c - \alpha r^2) + a \tag{4.125}
$$

$$
\beta^2 \equiv -a^2 - 2b > 0 \quad \text{gives} \quad u = \beta \tan\left(c - \frac{1}{2}\alpha r^2\right) + a \tag{4.126}
$$

$$
a^2 + 2b = 0 \quad \text{gives} \quad u = a + \frac{2}{c - r^2} \tag{4.127}
$$

but no further details are given in the original.

Finally, the ansätze

$$
u = ae^{br^2} \tag{4.128}
$$

$$
u = a + br^2 \tag{4.129}
$$

lead to expressions for p and ρ which are "too lengthy" or too complicated.

Kuchowicz (1972c) [67]

This paper uses isotropic coordinates, but seems to be based on a fallacy. The author appears to confuse the isotropic coordinate r_{iso} defined in (**??**) and the r used in Schwarzschild coordinates. He assumes that r_{iso} must vanish at the centre, but that r need not; whereas in fact r_0 must vanish.

Bludman 1973 [7]

This paper is concerned with the polytropic equation of state (**??**), and extends, and to some extent generalises and criticises, the work of Tooper [90]. Bludman shows that for a stiff fluid, a neutron star with a polytropic equation of state predicts the maximum mass to an accuracy of between 5-10%, and the maximum radius to between 10-20%.

Kuchowicz 1973a [68]

is mainly concerned with inequalities in isotropic coordinates. However, two solutions are given: one is possibly 1-parameter, and the other is shown to be unrealistic.

Kuchowicz 1973b [68]

gives an interesting addendum to (4.115) of [66]. He notes that the metric coefficients in Schwarzschild coordinates can be given in terms of u and v even though u and v are defined in (4.115) in terms of ν and μ in isotropic coordinates! The relevant equations are:

$$\begin{aligned} e^{-\lambda} &= 1 + q(u - v) \\ \nu(q) &= a + \frac{1}{2}\int v\, dq \end{aligned} \tag{4.130}$$

Thus an ansatz for $u(q)$ will produce $v(q)$ from (4.115) and hence the metric coefficients via (4.130). Since Scshwarzscild coordinates are now being used, we need to have these coefficients as functions of r and so need to find $q(r)$ or $r(q)$ from

$$r^2 = qb \exp \int (u - v)\, dq \tag{4.131}$$

Although this method looks simple, no solutions are given using it.

Adler 1974 [2]

Adler rediscovers and investigates (**??**), but without acknowledging the earlier work.

Krori and Borgohain (1974) [50]

This paper extends earlier work by Krori on composite stars, with a core described by the interior Schwarzschild solution, and two outer shells described by the Tolman IV, V and VI solutions. The models are examined to determine for which ranges of the parameters involved they are physical.

Adams and Cohen (1975) [1]

This paper gives three solutions, the first of which is another rediscovery of Wyman's regular solution (**??**), without acknowledgement. The second solution given is a duplication of solution (4.82) of Kuchowicz [57]. The third solution given violates (**??**) and (**??**). However, a significant point in this paper is that the authors compare solution (**??**) with a computer model, concluding that there is agreement with the gross features to within 15%.

The neutron star model they use is one due to Borner and Cohen [9], with a Bethe-Johnson equation of state. Table 1 is a reproduction of part of the table given in [1], but inverting the quantity in the third column. Adams & Cohen obtain their figure of 15% agreement by considering ρ_0 and p_0 as initial data and numerically calculating M, r_S and z_0 using (**??**). However, Collins [18] shows that extrapolation from central data outwards is not always a safe procedure. Because of this, and the fact that the parameters of solution (**??**) are given in terms of M and r_S, it seems instructive to calculate ρ_0, p_0 and z_0 from (**??**), given data on the surface from table 1. The results are displayed in table 2.

It can be seen that the average error is over 45%, a threefold increase in error over the method of Adams &

Table 1: Selected Extracts of Borner & Cohen Neutron Star Parameters as Given by Adams & Cohen

M $(10^{33}g)g$	R_S $(10^5 cm)$	ρ_0 $(10^{15}g)$	p_0 $(10^{35} dyn\, cm^{-2})$	ρ_0/p_0	z_0 $(e^{-\nu_0}-1)$
3.72	9.78	3.16	12.50	2.28	2.08
3.69	10.25	2.51	8.23	2.75	1.64
3.59	10.73	2.00	5.34	3.38	1.28
3.39	11.19	1.58	3.37	4.22	0.98
3.10	11.59	1.26	2.11	5.38	0.74
2.71	11.91	1.00	1.27	7.09	0.55

Table 2: Neutron Star Parameters from Wyman's solution. Percentage Errors are Given in Brackets.

M	R_S	ρ_0	p_0	z_0
3.72	9.78	3.16(-46)	12.50(-64)	2.08(-40)
3.69	10.25	2.51(-45)	8.23(-61)	1.64(-36)
3.59	10.73	2.00(-45)	5.34(-59)	1.28(-32)
3.39	11.19	1.58(-46)	3.37(-58)	0.98(-29)
3.10	11.59	1.26(-49)	2.11(-58)	0.74(-27)
2.71	11.91	1.00(-51)	1.27(-59)	0.55(-26)

Cohen. So their conclusion that their (or, in actual fact, Wyman's) solution "duplicates the gross features of a sophisticated computer model within 15%" is only true with their method of calculation.

Kuchowicz 1975 [69]

rediscovers and investigates Wyman's solution (**??**), without acknowledging the original.

Krori, Nandy and Bhattacharjee (1976) [51]

This paper presents a "solution" arising from an ansatz originally made by Tolman — the Tolman VII solution. However, the expression for the metric coefficient g_{00} is given incorrectly.

Collins (1977) [17]

Collins discusses the isothermal equation of state, and shows from mathematical considerations, and a physical assumption, that it is "the equation of state most relevant to relativistic stellar structure". He uses a result of Chandrasekhar [16] that, for Newtonian fluid spheres, a gamma-law or polytropic equation of state implies the existence of an homologous family of solutions, *ie* a set of solutions related by

$$R \to R_{new} = aR$$

$$\rho \to \rho_{new} = b\rho$$

$$m \to m_{new} = cm \tag{4.132}$$

with a,b and c constants. Collins reverses this argument and shows that the existence of an homologous family of solutions implies a gamma-law or polytropic equation of state.

Applying this reasoning to the relativistic fluid sphere, Collins shows that the only allowable transformations of R, ρ and m are the identity transformation, or (4.132) with

$$b = a^{-2}; c = a \tag{4.133}$$

This tighter form of (4.132) implies a gamma-law equation of state only. Thus mathematical considerations show that the existence of an homologous family of solutions, as defined by (4.132) and (4.133), for a fluid sphere necessitates a gamma-law equation of state.

The physical assumption that leads to the paper's claim that an isothermal fluid is "the most relevant..." is simply that homologous families of solutions do exist. However, no reason is given for why this should be so, other than "it seems reasonable".

The final section of the paper uses qualitative dynamical techniques to describe isothermal perfect fluid spheres in terms of the "homologous invariants" M/r and ρr^2. His conclusion is that there are only two possible realistic solutions, *viz* the Misner-Zapolsky solution (**??**) which is both geometrically and physically irregular at the centre, and another, undiscovered, solution with finite central density. However, since $p \propto \rho \propto r^{-2}$, the boundary of the sphere is at infinity. The corollary of this work would seem to be that the "reasonable" requirement of the existence of an homologous family of solutions is that no finite fluid sphere can exist at all!

Whitman (1977) [94]

Rediscovers Wyman's irregular solution (**??**), but does not acknowledge this.

Bayin (1978) [?]

This paper gives seven solutions, of which the author claims five are new. In fact only one is new, and that is a simple generalisation of a solution of Tolman's! There are also considerable misprints or errors in the paper as regards the numbering of sections and solutions. The first five solutions are in Schwarzschild coordinates.

Bayin's first solution is a rediscovery of Kuchowicz's solution (4.85), which Bayin acknowledges, and was also given by Heintzmann [41].

The second solution is the only new solution in the paper, and is a simple variation on the Tolman VII solution (**??**). The form of g_{00} for both solutions may be written as

$$e^{-2\lambda} = 1 + ar^2 + br^4 \tag{4.134}$$

Tolman assumes that the coefficients a and b are respectively negative and positive, while Bayin makes no such assumption. For this solution, ρ increases outwards, so the solution is not realistic.

The third solution is a rediscovery of a solution of Kuchowicz (4.78), but Bayin gives the forms of $e^{2\lambda}$, p and ρ, and shows that the solution is irregular.

Solution IV is a solution originally due to Heintzmann (**??**). Bayin shows that $p \to -\infty$ at the centre.

Solution V is the sixth (but not final) rediscovery of (**??**), though Bayin cites Adler [2], Adams & Cohen [1] and Kuchowicz [56] as references. In fact, the solution appeared in a later paper by Kuchowicz [69].

The final two solutions are given in isotropic coordinates. Solution VI had already been given by Kuchowicz (4.113), but Bayin shows that it is elementary, whereas Kuchowicz had claimed that it was not. However, Bayin's claim that it is realistic is erroneous. The pressure and density may be calculated to have the forms

$$\begin{aligned} p &= \frac{bC_0^2(b-2a)r^2 + 2C_0(b-a)}{BC_1^b} e^{-bC_0 r^2} \\ \rho &= \frac{-6bC_0 - b^2 C_0^2 r^2}{BC_1^b} e^{-bC_0 r^2} \end{aligned} \tag{4.135}$$

from which it can be derived that

$$\frac{dp}{d\rho} = \frac{C_0(2a-b)r^2 - 1}{C_0 b r^2 + 5} \tag{4.136}$$

whence it is easily seen that near the centre $v_s < 0$. Bayin also states that p and ρ decrease outwards, but (4.135) shows this is not the case.

Finally, Bayin's solution VII was first discovered by Kuchowicz [66] — it is given as (4.123). Fixing one of the constants, Bayin obtains an equation of state which he says is "an analog of a relativistic polytrope":

$$p = K\rho^n + \rho\frac{p_0}{\rho_0} \tag{4.137}$$

In conclusion, none of the solutions given in this paper are shown to be realistic and regular, and the paper as written contains many errors.

Ellis, Maartens and Nel (1978) [31]

In contrast with most work done on SSSPF solutions, this work studies a cosmological solution, in the context of providing an example of a cosmology which is isotropic around *a particular* observer, but not homogeneous. The authors divide the fluid into matter and radiation components (the latter with a $p = \frac{1}{3}\rho$ equation of state, the former with an equation of state $p = n\rho$) and show, among other things, that it is possible to discuss an infinite static sphere realistically.

Glass and Goldman (1978) [39]

Glass and Goldman produce a solution generating algorithm for use in isotropic coordinates. In contrast to other algorithms, such as those of Heintzmann [41] and Kuchowicz [66], their algorithm is guaranteed to produce solutions satisfying (??)–(??), (??) and (??) Starting from the metric (??), the authors define new variables

$$\begin{aligned} e(r) &\equiv \frac{rp'}{\rho + p} \\ f(r) &\equiv \frac{p}{\rho + 3p} \\ u(r) &\equiv r\nu' + \mu' \end{aligned} \tag{4.138}$$

and show that all functions can be written in terms of $u(r)$ and r:

$$2e^2 = 2u + u^2 - ru'$$

$$
\begin{aligned}
f &= \frac{e^2 - 2u - u^2}{2(e + eu + re')} \\
\nu' &= -\frac{e}{r} \\
\mu' &= r^{-1}(e(1-f) - 1 + \sqrt{1 + e^2(1+f^2) - 2rfe'}) \\
p &= \exp\left[\int \frac{e}{r}\left(\frac{1}{f} - 2\right) dr\right] \\
\rho &= p\left(\frac{1}{f} - 3\right)
\end{aligned}
\tag{4.139}
$$

The algorithm begins by choosing a form of $u(r)$ such that

$$
u = \mathrm{O}\left[r^2\left(l - \frac{1}{2}k^2 k_1 r^2 + \frac{1}{4}k^3 k_2 r^4\right)\right] \tag{4.140}
$$

and $u \geq 0$ as $r \to 0$ with a finite first maximum to the right of $r = 0$, with $k \neq 0$, $k_1 > \frac{23}{9}$ and k_2 such that

$$
25k_2^2 + 10k_2(12 + 17k_1) - (18 + 78k_1 + 197k_1^2 + 162k_1^3) > 0 \tag{4.141}
$$

Now, calculate e^2 from (4.139), and choose $e = -\sqrt{e^2}$. All other quantities may now be found from (4.139). The surface of the resulting sphere is given by

$$
r_S = -\frac{e^2}{u'} \tag{4.142}
$$

and the resulting solution satisfies (**??**)–(**??**), (**??**) and (**??**) and is also regular.

The authors give a two-parameter solution using the algorithm, by choosing

$$
\begin{aligned}
u(r) &= \frac{2r^2(1-r^2)}{F} \\
F(r) &\equiv \beta^2(2+\delta) - (\beta^2\delta + 1)r^2 + r^4
\end{aligned}
\tag{4.143}
$$

with β and δ constants, and obtain

$$
\begin{aligned}
e^\nu &= A_1 H^{\beta/\gamma} \\
e^\mu &= A_2 F^{-1/2} H^{-\alpha/2} \\
\frac{p}{p_0} &= \beta^2\left[2 + \delta - (3 + 2\delta r^2) + \delta r^4\right]\left(\frac{H}{H_0}\right)^\alpha F^{-1} \\
\frac{\rho}{\rho_0} &= \frac{J\beta}{3F(\beta - 1)}\left(\frac{H}{H_0}\right)^\alpha \\
r_S^2 &= 1 + \frac{3}{2\delta}\left(1 - \sqrt{1 + \frac{4}{9}\delta}\right)
\end{aligned}
\tag{4.144}
$$

with A_1 and A_2 constants, and the following definitions:

$$\begin{aligned} \gamma^2 &\equiv (\beta^2\delta - 1)^2 - 8\beta^2 \\ \alpha &\equiv \frac{\beta^2\delta + 2\beta - 1}{\gamma} \\ H(r) &\equiv \frac{2r^2 - (\beta^2\delta + 1 + \gamma)}{2r^2 - (\beta^2\delta + 1 - \gamma)} \\ J(r) &\equiv 3\beta(\beta-1)(2+\delta) + (1 + 6\beta\delta + 9\beta - \beta^2\delta)r^2 - 3(1+\beta\delta)r^4 \end{aligned} \tag{4.145}$$

The constraints on k_1 and k_2 lead to

$$\begin{aligned} \beta^2 &> \frac{16}{9} \\ (5\beta^2\delta - 7)^2 &> (18\beta)^2 \end{aligned} \tag{4.146}$$

while (4.144) also implies

$$\delta > -\frac{9}{4} \tag{4.147}$$

The authors also note that at the centre the equation of state is

$$\rho_0 = 3p_0(\beta - 1) \tag{4.148}$$

Whitman (1978) [95]

In this paper, Whitman studies the assumption $e^\nu = ar^m + br^n$ in Schwarzschild coordinates, recovering the results of Wyman [99].

Durgapal, Pande and Pandey (1979) [26]

This paper presents no new solutions, but extends the numerical work of Misner and Zapolsky [75] to stars with equation of state $p = n\rho$ and finite central density.

Durgapal and Rawat (1980) [28]

This paper performs a numerical analysis of a special case of the Tolman VII solution (which the authors do not acknowledge), given in [25]. They determine a maximum mass for a realistic neutron star of $3.34M_\odot$.

Kramer et al (1980) [47]

In §14.1 of their book on exact solutions to Einstein's field equations, Kramer et al give a concise review of SSSPF solutions. They give a table of 10 solutions, and point out correctly that most solutions are unphysical.

Korkina (1981) [45]

The paper first describes the physical regularity conditions at the centre of the fluid sphere in Schwarzschild coordinates. It proceeds to use these regularity conditions to determine two discrete classes of exact solutions which are guaranteed to be regular at the centre. Korkina makes an ansatz which amounts to the assumption that g_{00} should assume the form

$$e^{2\nu} = A(1+u^2)^n \tag{4.149}$$

where $u \equiv br$, with A, b and n constants. This assumption is also made in later work by Durgapal [21], and Durgapal *et al* [27]. Korkina shows that the pressure is expressible in terms of elementary functions provided the integral

$$I = \int \frac{(1+u^2)^{n-1}\,du}{u^3\,[1+(n+1)u^2]^{(1-n)/(1+n)}} \tag{4.150}$$

can be evaluated in terms of elementary functions. Korkina finds that this is so provided n is an integer (see also [21]), or $(1-n)/(1+n)$ is an integer (see also [27]). Korkina rejects the second class of solutions and points out (as does Durgapal later) that the first two solutions in this series are the Tolman IV solution (**??**), and Wyman's solution (**??**). Korkina rejects the second class of solutions as unrealistic, stating that n must always be negative, thereby leading to an unphysical equation of state. However, this is not the case, as shown by Durgapal *et al* [27]. As an example of a new solution, Korkina gives the case when $n = 3$, namely

Kork ($n = 3$)

$$\begin{aligned} e^{2\nu} &= A(1+u^2)^3 \\ e^{-2\lambda} &= 1 - \frac{3u^2}{2(1+u^2)} - \frac{Cu^2}{2(1+u^2)(1+4u^2)^{1/2}} \end{aligned} \tag{4.151}$$

with complicated expressions for p and ρ. This solution is matched to the exterior Schwarzschild solution, and the value of the surface redshift for a central equation of state $p = \rho/3$ is calculated.

Ibañez and Sanz (1981) [43]

The authors present two "new" SSSPF solutions, the second of which is obtained from the first by using Heintzmann's generating technique [41]. In fact, neither solution is new, the first (an isothermal fluid) being another rediscovery of the Misner-Zapolsky solution (**??**), while their second solution is just the Tolman V solution (**??**). If we use the subscripts IS and T to represent the constants in the Ibanez-Sanz and Tolman papers respectively, the identification is

$$n_{IS} = \frac{n_T}{2-n_T};\, B_T = 1 \tag{4.152}$$

The paper therefore contains no new solutions.

Durgapal (1982) [21]

This paper largely repeats work done earlier by Korkina [45]. No new classes of solutions are given, though Durgapal gives explicit forms for the next two solutions in Korkina's first class, and analyses the physical characteristics of the solutions in great detail. Durgapal assumes

$$e^{2\nu} = A(1+x)^n \tag{4.153}$$

where $x \equiv Cr^2$, A and C are constants, and shows that Einstein's equations can be solved explicitly if the integral

$$I = \int \frac{(1+x)^{n-1}\,dx}{x^2\,[1+(n+1)x]^{(n-1)/(n+1)}} \tag{4.154}$$

can be evaluated in a closed form. This is entirely equivalent to Korkina's approach. The author states that this is so for any (positive integer) value of n, and, although the expression given in equation (17) of the original paper is incorrect, this is indeed the case. The solutions for $n = 1, \ldots, 5$ are given, along with equations for the pressures, densities and sound speeds for all these solutions. The first two solutions in this series, as the author (and Korkina) indicates, are the Tolman IV solution (**??**), and Wyman's solution (**??**). The third one, though not acknowledged, is the solution explicitly presented by Korkina. The two "new" solutions explicitly presented are:

Kork $(n = 4)$

$$e^{2\nu} = A(1+x)^4 \tag{4.155}$$

$$e^{-2\lambda} = \frac{7 - 10x - x^2}{7(1+x)^2} + \frac{Kx}{(1+x)^2(1+5x)^{2/5}} \tag{4.156}$$

Kork $(n = 5)$

$$e^{2\nu} = A(1+x)^5 \tag{4.157}$$

$$e^{-2\lambda} = \frac{1 - 309x - 54x^2 - 8x^3}{112(1+x)^3} + \frac{Kx}{(1+x)^3(1+6x)^{1/3}} \tag{4.158}$$

In the above solutions, $x \equiv Cr^2$, with A, K and C arbitrary constants. The solutions are investigated for realism, and the author distinguishes between two properties of the sound speed: that the solution should have $v_S < 1$ throughout, and that the central value of v_S should be a maximum (ie, that the sound speed decreases monotonically with density). In terms of the parameter $u = m(r)/r$, these are satisfied for the new solutions provided that

$$n = 3 \quad u \lesssim 0.35 \quad u \geq 0.375 \tag{4.159}$$

$$n = 4 \quad u \lesssim 0.313 \quad u \geq 0.30 \tag{4.160}$$

$$n = 5 \quad u \lesssim 0.265 \quad \text{no restriction} \tag{4.161}$$

It should be noted that there is one small misprint in the equations in this paper: the form of $dp/d\rho$ for $n = 3$ should read

$$\frac{dp}{d\rho} = \frac{9 - 3x - 2K(1 - x - 14x^2)(1+4x)^{-5/2}}{5 + x - 2K(5 + 23x + 30x^2)(1+4x)^{-5/2}} \tag{4.162}$$

Collins (1983) [18]

Collins uses results of an earlier paper [17] to comment on, and extend, the work of Ellis et al [31]. Collins points out that for certain equations of state, the central conditions do not uniquely determine the geometry for $R > 0$. This is important for numerically integrated models, which are usually determined by the central conditions.

Durgapal and Bannerji (1983) [22]

This paper rediscovers the solution of Buchdahl (**??**), which is not acknowledged. The authors investigate the solution and show that the solution is regular (which is easily seen from the central forms of g_{00} and g_{11}), and realistic, provided that the parameter

$$u \equiv M/r_S \lesssim 0.3434 \tag{4.163}$$

Whitman (1983) [96]

Whitman's paper has some similarities to Glass & Goldman, in that he gives a prescription for obtaining solutions which satisfy (??)–(??), (??) and (??), but uses Schwarzschild rather than isotropic coordinates. He also presents a solution generating method which strongly resembles that of Heintzmann [41], but this fact is not acknowledged. However, all Whitman's solutions are given in terms of a Heun function, which, in special cases, reduces to hypergeometric functions. Whitman concludes that "it is doubtful whether much simpler solutions can be found" using his method.

Durgapal, Pande and Phuloria (1984) [27]

The first part of this paper simply reiterates the earlier work of Kuchowicz [58], Leibovitz [71] and Korkina [45], with the authors apparently unaware of this earlier work. The authors then proceed to study the consequences of the ansatz proposed in [45] and [21] more carefully, in particular the conditions on n under which the integral (4.154) can be evaluated in closed form. They rediscover the result of Korkina, namely that provided

$$k \equiv \frac{1+n}{1-n} \in Z, \tag{4.164}$$

this can be achieved. They do not make the criticism which Korkina makes of this class, namely that the equation of state is unrealistic. The solution when $n = \frac{1}{2}$ is given as

DPP1

$$\begin{aligned}
e^{2\nu} &= \frac{B}{1-x} \\
e^{-2\lambda} &= \frac{4(1-x^2)(4-4x-x^2)+16Ax(1-x)^{5/2}}{(2-x)^4} \\
\frac{p}{C} &= \frac{16A(1-x)^{3/2}-12x+12x^2-x^3}{(2-x)^4} \\
\frac{\rho}{C} &= \frac{96-120x-60x^2+909x^3-5x^4-16A(6-11x)(1-x)^{3/2}}{(2-x)^4}
\end{aligned} \tag{4.165}$$

where $x \equiv Cr^2$, and A, B and C are constants. The authors do not pursue the realism of this solution further, because of the few adjustable parameters involved, but instead proceed to describe the generalisation of the above (particular) solutions to solutions with one more parameter. A detailed study of a particular case of Tolman VII (see also [28]) is then given.

Durgapal & Fuloria (1985) [23]

This paper presents a model for a "superdense" star, with the solution being presented as

$$e^{2\nu} = B(1+x)^2 + A\left[(7+3x)(7-10x-x^2)^{1/2} + 4W(x)(1+x)^2\right]$$

$$
\begin{aligned}
e^{-2\lambda} &= 1 - \frac{8x(3+x)}{7(1+x)^3} \\
\rho &= 8C\frac{9+2x+x^2}{7(1+x)^3} \\
W(x) &\equiv \ln\left[\frac{(3-x)(7-10x-x^2)^{1/2}}{(1+x)}\right]
\end{aligned}
\tag{4.166}
$$

where A, B and C are constants, and $x \equiv Cr^2$. Although the authors show that the solution has a realistic value for the sound speed if $u \equiv m/r_S \lesssim 0.4265$, they do not investigate the ratio p/ρ for realism, instead allowing it to range from 0 to ∞.

Berger & Hojman (1987) [6]

This paper is misleading for two reasons. Firstly, in the abstract of this paper, the authors claim that "the gravitational field equations for a spherically symmetric perfect fluid are completely solved". In fact, the authors only look at the static case, though they nowhere state this explicitly. Secondly, the authors merely use the Gaussian coordinate system (**??**) in which the static, spherically-symmetric metric may be written

$$ds^2 = e^{2\nu(r)}dt^2 - dr^2 - R^2(r)d\Omega^2 \tag{4.167}$$

and present equations for the metric coefficients, density and pressure which involve quadratures. This is clearly no advance on Tolman's seminal paper!! The authors proceed to describe the interior and exterior Schwarzschild solutions in their coordinate system, and proceed to state that their method produces "new explicit solutions", none of which are unfortunately shown.

Dourah & Ray (1987) [20]

In this paper the authors claim to produce a realistic SSSPF solution by adopting an ansatz for the metric component g_{11} of the form

$$e^{2\lambda} = 1 + Cr^2, \tag{4.168}$$

where C is a constant. The resulting solution would be regular at the centre, but the authors solve Einstein's equations incorrectly. The correct solution is given by Finch and Skea [33].

Finch (1987) [32]

Finch & Skea (1989) [33]

This paper points out that the solution given by Duorah and Ray [20] is incorrect, and presents the correct solution, which is given by

$$
\begin{aligned}
e^{\nu} &= A\left[(C_2 - C_1 v)\cos(v) + (C_1 + C_2 v)\sin(v)\right] \\
e^{2\lambda} &= v \\
v &\equiv \sqrt{1 + Cr^2}
\end{aligned}
\tag{4.169}
$$

and A, C, C_1 and C_2 are arbitrary constants. The authors give a rather complicated equation of state, and show that the solution is realistic for a range of values of the parameter β, such that

$$0.218 \lesssim \beta \equiv \frac{C_1}{C_2} \lesssim 9.297. \tag{4.170}$$

Finally, the authors compare their solution with data for a model of a neutron star based on a description of a neutron fluid due to Walecka [92]. The predicted central densities of the neutron star are shown to agree with the values given in Arnett & Bowers [4] to "within an average of 12%."

Table 3: Ansatze used for g_{00} in Schwarzschild coordinates

Quantity	Ansatz	Equation no(s)
$e^{2\nu}$	$1-\frac{2M}{r}$	(??)
$e^{2\nu}$	$a+br^2$	(??)
e^{ν}	ar^n	(??)
e^{ν}	$a+br^2$	(??), (4.96)
e^{ν}	$a+br^n$	(4.94)
e^{ν}	$ar^{1-n}+br^{1+n}$	(??), (??)
$e^{2\nu}$	$(a+br^2)^3$	(4.100), (4.151)
$e^{2\nu}$	$(a+br^2)^4$	(4.156)
$e^{2\nu}$	$(a+br^2)^5$	(4.158)
$e^{2\nu}$	$\frac{a}{1-br^2}$	(4.165)
e^{ν}	$ar+br\ln r$	(4.67)

5 Tables

The tables in this section describe the ansätze used to produce most of the solutions in section §??. They are arranged in a manner which will hopefully allow the reader to locate quickly whether or not their particular assumptions have been previously considered in the literature. Two sets of tables are provided: one for Schwarzschild coordinates, the other for isotropic coordinates. The other coordinate systems in use occur so rarely that it was thought to be superfluous to include them here, when the relevant papers may be tracked down by studying the section on coordinate systems in §??.

For both coordinate systems, ansätze have been grouped into four categories: forms of g_{00}, g_{11}, ρ and a miscellaneous category. In constructing these tables, due consideration has had to be given as to what may reasonably be called an 'ansatz'. Some solutions may have simple forms for more than one of the categories described above, in which case the solution has been included in more than one category. Solutions where there is no obvious simple ansatz are rare, but not non-existent — see (??) — and have been excluded from these tables. No indication is given of whether or not the solutions to an ansatz have been discovered in complete generality: in fact, in many cases the original authors (and ourselves) have excluded extra constants because they either render the solution unphysical, or because they are superfluous. The reader is advised to refer to the original papers *always*.

Table 4: Ansatze used for g_{11} in Schwarzschild coordinates

Quantity	Ansatz	Equation no(s)
$e^{-2\lambda}$	a	(??), (4.57), (4.67)
$e^{-2\lambda}$	ar^n	(4.70)
$e^{-2\lambda}$	$1+\frac{a}{r}$	(??), (4.89)
$e^{-2\lambda}$	$1-ar^2$	(??)
$e^{-2\lambda}$	$1-ar^4$	(??)
$e^{-2\lambda}$	$1-ar^n$	(??)
$e^{-2\lambda}$	$a+br^2$	(4.58)
$e^{-2\lambda}$	$a-br^n$	(??)
$e^{-2\lambda}$	$1-ar^2+br^{-1}$	(??), (??)
$e^{-2\lambda}$	$1-ar^2+b^2r^4$	(??)
$e^{2\lambda}$	$1+ar^2$	(4.168), (4.169)

Table 5: Ansatze used for ρ in Schwarzschild coordinates

ρ	Equation no(s)
ρ_0	(??), (??), (??)
$\frac{1}{2r^2}$	(4.67)
ar^{-2}	(??), (4.57)
ar^n	(??)
$a+br^{-2}$	(4.58)
$a-br^2$	(??), (??)

Table 6: Other Ansatze used in Schwarzschild coordinates

Quantity	Ansatz	Equation no(s)
$e^{\nu+\lambda}$	ar^b	(??), (??)
$e^{-2(\nu+\lambda)}$	$a-br^2$	(??)
p	$n\rho$	(??), (??)
$\rho+3p$	a	(4.92)

Table 7: Ansatze used for g_{00} in Isotropic coordinates

Quantity	Ansatz	Equation no(s)
e^{ν}	$\frac{r-a}{r+a}$	(??)

Table 8: Ansatze used for g_{11} in Isotropic coordinates

Quantity	Ansatz	Equation no(s)
e^{μ}	$\left(1+\frac{m}{2r}\right)^2$	(??)
$e^{-2\mu}$	$ar^{1+n}+br^{1-n}$	(??)
$e^{2\mu}$	ar^b	(??)

6 Interesting Solutions

A question which is unfortunately seldom asked in any of the papers reviewed is this: What is the point of having these closed form solutions? What is their interest? In one sense they are obviously of interest, since so much mental energy has been expended on producing almost 100 papers which try to find these solutions! However, our definition of an "interesting solution" will be much more stringent. We suggest that an interesting solution should fulfil the following four criteria:

(i) The solution should be physically and geometrically regular in the sense of §??, ie the solution should match to the Schwarzschild exterior solution, it should be possible to construct a locally Lorentzian frame everywhere, and ρ_0 and p_0 should be finite.

(ii) The solution should be physically *realistic*, in the sense of (??)–(??), (??) and (??) being fulfilled up to the sphere's boundary.

(iii) The solution should have a minimum amount of flexibility. In practice, a solution will satisfy this criterion if it has at least two free constants (ie, is at least two-parameter) after all regularity conditions have been satisfied.

(iv) The solution should be realistic in a range of parameters which typify an astrophysical situation. In practice, this will be taken to mean a neutron star.

Two further complementary characteristics of a solution will also be of interest: simplicity and flexibility. Though simplicity is a subjective quality, it can be given some rigour by the following definition: a *simple* solution is one which can be shown to be realistic by *non-numerical* means. A flexible solution will be defined

Table 9: Ansatze used for ρ in Isotropic coordinates

ρ	Equation no(s)
0	(??)

as one which, though not simple, can duplicate a numerical solution (ie a neutron fluid model) to a fair degree of accuracy. Thus a 3-parameter solution which fulfils criteria (i) to (iv) outlined above, and is reasonably simple, would also be of interest.

There are only eight solutions (or classes of solutions) which satisfy our criteria for an interesting solution. They are: Tolman IV (**??**); Wyman's oft-rediscovered solution (**??**); Buchdahl's solution of 1959 (**??**), subsequently studied by Durgapal and Bannerji [22]; Buchdahl's solution of 1967 (**??**); Heintzmann's regular solution (4.96); Glass and Goldman's solution (4.144) and the solution of Finch and Skea (4.169).

However, if our criteria were interpreted strictly, most of these solutions would be disqualified. For instance, it could be demanded that the equation of state should be expressible in terms of ρ an p alone, and not parametrically in terms of r. Only the Tolman IV solution, Buchdahl's 1967 solution, and the solution of Finch and Skea have equations of state expressible as $p = p(\rho)$. In addition, Heintzmann's solution (4.96) does not strictly satisfy criterion (ii), since it is not shown that (**??**) is satisfied: it is only included because of the heuristic argument following the presentation of the solution. Buchdahl's solution (**??**) would probably be disqualified on account of criterion iv: studying the Borner-Cohen neutron star parameters given by Adams and Cohen in table 1, a comparison of the values of $\frac{p_0}{\rho_0}$ with (**??**) shows that the solution is hardly relevant.

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