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I. BACKGROUND

This section gathers together information on isolated static spherically symmetric spacetimes with an origin that are exact solutions of Einstein's equations for a perfect fluid decomposition. (For convenience we label such a spacetime as **M**.) By isolated we mean that the fluid smoothly joins onto a vacuum exterior across a timelike boundary surface (Σ) of finite surface area. For definiteness, consider the coordinate labels (r, θ, ϕ, t). The metric can be written in adapted form [4] as

$$ds^2 = B(r)dr^2 + C(r)r^2 d\Omega^2 - A(r)dt^2 \tag{1}$$

where $d\Omega^2$ is the metric of a unit sphere $d\theta^2 + sin^2(\theta)d\phi^2$. There are only two free functions. The most common representations are $C(r) = B(r)$ for *isotropic coordinates* and $C(r) = 1$ for *curvature coordinates*. Associated with **M** is the effective gravitational mass (M), taken here to be defined by

$$M \equiv \frac{g_{\theta\theta}^{3/2}}{2} R_{\theta\phi}{}^{\theta\phi}. \tag{2}$$

The invariant properties of M were first explored by Hernandez and Misner [5]. Within curvature coordinates **M** takes the form

$$ds^2 = \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2. \tag{3}$$

A. Fluid decomposition and Einstein's equations.

For concreteness let us take the representation (3) with the coordinates (r, θ, ϕ, t) taken to be comoving in the sense that the mathematical fluid streamlines are given by $u^a = e^{-\Phi(r)}\delta^a_t$. The fluid decomposition is taken to be [6]

$$T^a_b = (\rho(r) + p(r))u^a u_b + p(r)\delta^a_b. \tag{4}$$

Solving for $\Phi'(r)$ ($' \equiv \frac{d}{dr}$) (for example from the r-component of the conservation equations and Einstein's equations, $\nabla_a T^a_r = 0$ and $G^r_r = 8\pi p(r)$) we obtain the Tolman -Oppenheimer-Volkoff (T-OV) equation [7]

$$\Phi'(r) = \frac{-p'(r)}{\rho(r) + p(r)} = \frac{M(r) + 4\pi p(r) r^3}{r(r - 2M(r))}, \tag{5}$$

where, from the t component of the Einstein equations ($G^t_t = -8\pi\rho(r)$),

$$M(r) = 4\pi \int_0^r x^2 \rho(x) dx. \tag{6}$$

Detailed properties of (5) were slow to arrive. For example, only relatively recently has it been shown that there exits a unique global solution to (5) for every finite non-vanishing $p(0)$ [8].

B. Regularity.

Regularity conditions can be stated in terms of contractions and traces of the Riemann tensor. These conditions impose restrictions on the physics and of course also on the coordinate representation of the spacetime. We expect exact solutions to be exhibited in regular coordinates. Coordinate regularity conditions for the metric (1) at the origin ($r = 0$) are known [9]. Here we review the regularity conditions stated in terms of physical quantities. With spherical symmetry, there are at most three independent Ricci invariants and one independent Weyl invariant [10]. For a perfect fluid there are but two independent Ricci invariants which can be taken to be the Ricci scalar

$$R = 8\pi(\rho(r) - 3p(r)) \tag{7}$$

and the first Ricci invariant (taken here to be $r_1 \equiv \frac{1}{4} S_\alpha^\beta S_\beta^\alpha$, S_β^α the trace-free Ricci tensor)

$$r_1 = 3(2\pi(\rho(r) + p(r)))^2. \tag{8}$$

The independent Weyl invariant can be taken to be $w_1 \equiv C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$, $C_{\alpha\beta\gamma\delta}$ the Weyl tensor, and for a perfect fluid (in the coordinates of (3)) w_1 is given by

$$w_1 = 48\pi(\frac{M(r)}{r^3} - \frac{4\pi\rho(r)}{3})^2. \tag{9}$$

For static spherically symmetric perfect fluids the Riemann invariants give no further information [11]. At the centre of symmetry ($r = 0$) the regularity of the Ricci invariants requires that $\rho(0)$ and $p(0)$ be finite. The regularity of the Weyl invariant requires that $M(r)$ is C^3 at $r = 0$ with $M(0) = M(0)^{'} = M(0)^{''} = 0$ and $M(0)^{'''} = 8\pi\rho(0)$. The centre of symmetry is of course conformally flat. These properties of $M(r)$ follow immediately from (6). In summary, for a static spherically symmetric perfect fluid, finite $\rho(0)$ and $p(0)$ guarantees the regularity of all Riemann invariants at the centre of symmetry.

C. Barotropic relation.

From the symmetry of **M** we have $\rho = \rho(r)$ and $p = p(r)$. From (5) we know that $p(r)$ is maximal at $r = 0$ and monotone decreasing to the boundary at $p(r_\Sigma) = 0$. From the inverse function theorem then $r = r(p)$ and so, for every **M**, there exists, in a strict sense, a "barotropic" relation

$$\rho = \rho(p). \tag{10}$$

This is an existence statement only as in general (10) is not available explicitly. The relation

$$p = p(\rho) \tag{11}$$

always exists (at least piecewise, and one expects that the cases that are only piecewise are also unstable). here we do not refer to (11) as an "equation of state" as **M** contains neither thermodynamics nor details on the material being modeled and so relation (11) could in fact reflect many different "equations of state".

D. Causality.

Now consider a perturbed space $\mathbf{M}'$. $\mathbf{M}'$ represents **M** with a disturbance ("sound wave") propagating in it. (There is no disturbance in **M**, it is static.) Assumptions have to be made on the construction of $\mathbf{M}'$. We are concerned with (Eulerian) perturbations in $\mathbf{M}'$ and need to decompose p with respect to perturbed fluid variables. Take

$$p = p(\rho, \psi_i), \tag{12}$$

where the ψ_i represent these variables (e.g. entropy per particle (S), chemical composition, etc) so that

$$\delta p = \frac{\partial p}{\partial \rho}\delta\rho + \Sigma_i \frac{\partial p}{\partial \psi_i}\delta\psi_i. \tag{13}$$

Assumptions on the construction of $\mathbf{M}'$ amount to assumptions on ψ_i. The simplest assumption is that the transition from **M** to $\mathbf{M}'$ preserves the perfect fluid form, $\psi_i = S$ only, and the transition is *adiabatic* in the sense that $\delta S = 0$. One writes

$$\frac{\partial p}{\partial \rho} \equiv v_s^2 \tag{14}$$

and if v_s is evaluated from $\rho(r)$ and $p(r)$ of **M** (for $\frac{d\rho}{dr} \neq 0$) it is called the "adiabatic sound speed" of **M**. As a consequence, one has the (formal) causality requirement [12]

$$v_s^2 < 1. \tag{15}$$

Whereas it could be argued that "small" perturbations of **M** should be adiabatic and so v_s should be a reasonable approximation to the physical speed of sound in **M**, this argument cannot be quantified. Too little is actually known about **M**. All that we are given is a phenomenological decomposition of the energy momentum tensor with no information on the material **M** is supposed to model. The actual physical speed of sound depends on the detailed structure of this material. For example, in the case of the familiar Schwarzschild interior solution, $\rho =$ const. and use of v_s would suggest an *infinite* adiabatic sound speed (the incompressible limit). Yet, even in this extreme case, it could be argued that the adiabatic sound speed is inappropriate. The distribution $\rho =$ const. may in fact model an object with a (contrived) composition variation rendering v_s meaningless as regards the speed of sound [13]. In a larger context, **M** could be viewed as a stable equilibrium state in dissipative relativistic kinetic theory. The Israel-Stewart theory has replaced the Eckart and Landau-Lifshitz formulations since only the former allows causal stable equilibrium states. Within this theory, available evidence suggests that (15) may not in fact strong enough when the adiabatic sound speed is an appropriate measure [14]. Here we simply record v_s for **M** [15].

E. Junction conditions.

We review the junction of a static perfect fluid onto vacuum by way of the Darmois - Israel conditions [16]. The exterior Schwarzschild vacuum line element, in terms of exterior curvature coordinates $(\mathsf{r},\theta,\phi,T)$, is given by

$$ds^2 = \frac{d\mathsf{r}^2}{1-\frac{2m}{\mathsf{r}}} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2) - (1-\frac{2m}{\mathsf{r}})dT^2. \tag{16}$$

The interior line element, in terms of interior curvature coordinates (r,θ,ϕ,t), is taken to be (3). The continuity of the first fundamental form (intrinsic metric) associated with Σ ensures that the continuity of θ and ϕ in metrics (3) and (16) is allowed and that the history of the boundary is given by [17]

$$r_\Sigma = \mathsf{r}_\Sigma > 2m. \tag{17}$$

In terms of intrinsic coordinates (τ,θ,ϕ) (τ the proper time on Σ) , the continuity of the extrinsic curvature component $K_{\tau\tau}$ along with the T-OV equation (5) gives

$$p(r_\Sigma) = 0. \tag{18}$$

We take (18) (with (17)) as the definition of the boundary Σ. The continuity of the extrinsic curvature components $K_{\theta\theta}$ and $K_{\phi\phi}$, along with (17), gives

$$M(r_\Sigma) = m. \tag{19}$$

The correct rigging of the 4-normals to Σ (interior to exterior) is verified by the continuity of the trace of the extrinsic curvature across Σ. Equations (17), (18) and (19) constitute the solution to the junction problem for the case of a static spherically symmetric perfect fluid joined onto vacuum. Although exhibited in curvature coordinates, these conditions are manifestly invariant.

Junction conditions are frequently confused with gauge conditions [18]. To further explain this point, it is useful to explore the restrictions imposed by the *assumption* that the coordinates used in (3) and (16) are admissible in the Lichnerowicz sense (*i.e.* there is no distinction between (t,T) and (r,r) and the 4-metric components are C^1 at Σ). In addition to the previous conditions, when applied to (t,T) the Lichnerowicz conditions give

$$e^{2\Phi(r_\Sigma)} = 1 - 2\frac{m}{\mathsf{r}_\Sigma}, \tag{20}$$

and when applied to (r,r) they give

$$M^{'}(r_\Sigma) = \rho(r_\Sigma) = 0. \tag{21}$$

Whereas (20) is clearly a forced choice of gauge, (21) is an unnecessary, albeit sometimes convenient, physical restriction.

F. Parameters.

Ideally, the physical significance of all elements of an exact solution should be clear. A static spherically symmetric metric admits the following 5 obvious constants of invariant physical significance: At the centre $p(0)$ and $\rho(0)$, and at the boundary r_Σ (defined by (18) (with (17)), m (defined by (19)) and $\rho(r_\Sigma)$ (defines the jump in ρ at r_Σ). Note that whereas m is *evaluated*, knowing r_Σ, r_Σ is solved for via $p(r_\Sigma) = 0$. In general r_Σ (and hence m) is not available analytically. Various combinations of these constants can of course be used, for example $\alpha \equiv r_\Sigma/m$. Further constants come from gradients of ρ [19]. The gradients of p are not independent as they are governed by (5). In summary then we have the following sets of constants: For derivatives of the metric to second order,

$$\mathbf{C} \equiv (p(0), \rho(0), r_\Sigma, m, \rho(r_\Sigma)) \tag{22}$$

and for derivatives of the metric to order $a > 2$

$$\mathbf{D} \equiv (\rho^{'}(0), ..., \rho^{2-a}(0), \rho^{'}(r_\Sigma), ..., \rho^{2-a}(r_\Sigma)). \tag{23}$$

If the metric is C^n then there are a total of $2n+1$ obvious constants. Of these, only 4 ($p(0), \rho(0), r_\Sigma$ and m) need to be non-zero. If an exact solution has more than $2n+1$ "parameters" in it, the individual *physical* significance of all these parameters is not a priori clear. In general, exact solutions, by their very nature, do not fall into the idealization envisioned here. That is, one can expect solutions to contain constants which do *not* offer an obvious physical interpretation [20].

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[1] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980).

[2] A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997).

[3] M. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395 (1998) (gr-qc/9809013).

[4] The associated Killing vectors are $\xi_1^\alpha = (0, sin(\phi), cot(\theta)cos(\phi), 0)$, $\xi_2^\alpha = (0, -cos(\phi), cot(\theta)sin(\phi), 0)$, $\xi_3^\alpha = (0,0,1,0)$ and $\xi_4^\alpha = (0,0,0,1)$. A useful reference is H. Takeno *The Theory of Spherically Symmetric Space-Times* in Scientific Reports of the Research Institute for Theoretical Physics, Hiroshima University No. 5 (1966).

[5] See W. C. Hernandez and C. W. Misner, Astrophys. J. **143**, 452 (1965). See also E. Poisson and W. Israel, Phys. Rev D **41**, 1796 (1990), T. Zannias, Phys. Rev. D **41**, 3252 (1990) and S. Hayward Phys.Rev. D **53**, 1938 (1996) (gr-qc/9408002).

[6] We do not include the cosmological constant explicitly in the present discussion.

[7] It is a curious fact that (5) has become known as the "Tolman -Oppenheimer-Volkoff" equation. Oppenheimer and Volkoff (J. R. Oppenheimer and G. Volkoff, Phys Rev. **55**, 374 (1939)) quote directly (to the page) from Tolman's book (R. C. Tolman *Relativity Thermodynamics and Cosmology* (Clarendon Press, Oxford 1934)) and Tolman himself makes no particular note of writing down Einstein's equations for a spherical static perfect fluid.

[8] T.W. Baumgarte and A.D. Rendall, Class. Quantum Grav. **10**, 327 (1993). See also M. Mars, M.M. Martín-Prats and J. M. Senovilla, Phys. Lett. **A**, 218, 147 (1996).

[9] For the metric (1) the coordinate conditions are as follows: $A(0)$ is a finite non-zero constant that fixes the gauge in t, $B(0)$ and $C(0)$ are finite non-zero constants (= 1 in curvature coordinates) and $A^{'}(0) = B^{'}(0) = C^{'}(0) = 0$. See B. Schmidt, *Kugelsymmetrische statische Materielösungen der Einsteinschen Feldgleichungen* (Diplomarbeit,Hamburg University) (1966), G.F.R. Ellis, R. Maartens and S.D. Nel, Mon. Not. Roy. Astron. Soc. **184**, 439 (1978) and K. Lake and P. Musgrave, General Relativity and Gravitation **26**, 917(1994). The conditions used in [3] were in fact too strong for the case of isotropic coordinates. All cases in isotropic coordinates were reexamined by S. Suyu (S. Suyu, *Physical acceptability of isolated, static spherically symmetric, perfect fluid solutions of Einstein's Equations*, Undergraduate Thesis, Queen's University (2001)). The results are summarized at `http://grtensor.phy.queensu.ca/solutions/` There were two significant changes, for **Buch3** and **P-S2** (in the notation of [3]). These solutions are discussed in this work.

[10] D. Pollney, N. Pelavas, P. Musgrave and K. Lake 1998, Computer Physics Communications **115**, 381 (1998).

[11] Equations (7), (8) and (9) could, of course, be used to define ρ, p and M invariantly.

[12] There has been a suggestion that $v_s^2 > 1$ is not necessary acausal, in the context of a particular classical model of matter with $p > \rho$ (G. Caporaso and K. Brecher, Phys. Rev D **20**, 1823 (1979)). This has not met with universal acceptance (see E. N. Glass, Phys. Rev. D **28**, 2693 (1983) and reponse by G. Caporaso and K. Brecher, Phys. Rev D **28**, 2694 (1983)) nor subsequent detailed justification.

[13] C.W. Misner, K. S. Thorne and J. A. Wheeler *Gravitation* (Freeman, San Francisco 1973).

[14] For example, recently Olson (T. Olson, Phys Rev C **63**, 015802 (2002)) has found that for a non-interacting degenerate Fermi gas in the low temperature limit $v_s^2 \leq \frac{\rho - \frac{p}{3}}{\rho + p}$. The degree to which the non-interacting assumption affects this limit is not clear.

[15] In the compilation [3] the condition $v_s^2 > 1$ was arrived at only for two solutions, one of which was the Schwarzschild interior solution.

[16] In general relativity the junction conditions approprite to a non-null boundary surface are the continuity of the first and second fundamental forms of the boundary. Following the pioneering work of Lanczos (K. Lanczos, Annalen Der Physik **74**, 518 (1924)) Darmois (G. Darmois, *Mémorial de Sciences Mathématiques, Fascicule XXV, "Les equations de la gravitation einsteinienne"* (1927)) set these conditions on a firm foundation. However, it was not until the influential paper by Israel (W. Israel, Nuovo Cimento 44B, 1 (see also 49B, 463) (1966)) that these conditions became well known. Israel

was interested primarily in surface layers (characterized by discontinuities in the extrinsic curvature) and was not aware of Darmois' paper. In special ("admissible") coordinates these conditions are equivalent to the continuity of the metric and its first derivative, conditions usually attributed to Lichnerowicz (a student of Darmois) (A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Paris, Masson, 1955)). It is interesting to note that Darmois was well aware of the limitations of "admissible" coordinates. A review (with unusual conventions) is given in [13]. The junction conditions can now be executed by computer algebra systems. See P. Musgrave and K. Lake, Class. Quantum Grav. **13**, 1885 (1996) (gr-qc/9510052).

[17] The well known Buchdahl limit $r_\Sigma > \frac{9}{4}m$ (H.A. Buchdahl, Phys. Rev. **116**, 1027 (1959)) is robust (J. Guven and N. O. Murchadha, Phys. Rev. D **60**, 084020 (1999) (gr-qc/9903067)). In the case of a thin shell surrounding dust (which volates the continuity conditions used in this paper) Bondi (H. Bondi, Proc. R. Soc. A **282** 303 (1964); and also *Lectures on General Relativity* Brandeis Summer Institute in Theoretical Physics, Volume One (Prentice Hall, New Jersey, 1964)) has achieved a limit of $r_\Sigma > \frac{1}{6\sqrt{2}-8}m$. For thin shells one can in fact obtain $r_\Sigma > 2m$ but not for physically relevant situations (see M. Ishak and K. Lake, Phys. Rev. D **65**, 044011 (gr-qc/0108058)).

[18] For example, the junction of the Schwarzschild interior and exterior solutions in [1] (see page 164) suffers from gauge fixing. The correct condition is $3\sqrt{1-\frac{r_\Sigma^2}{R^2}} = \frac{a}{b}$. The stated conditions include the choice of gauge $2b = 1$.

[19] To make the gradients invariantly defined let n^α be the outward radial normal to the fluid velocity u^α and define the gradient $n^\alpha \nabla_\alpha \rho \equiv \nabla\rho(r)$. In curvature coordinates then we have the further constants $\nabla\rho(0) \equiv \rho^{'}(0)$ and $\nabla\rho(r_\Sigma) \equiv \sqrt{1-2\frac{m}{r_\Sigma}}\rho^{'}(r_\Sigma)$. Going to the next order we get $\rho^{''}(0)$ and $\rho^{''}(r_\Sigma)$ and so on.

[20] Other, less obvious, parameters are possible. An extremal point in the potential impact parameter B provides one example.